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On some aspects of the theory of DQ-modules

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Contents

Introduction

A large part of my work is concerned with the study of categories of Deformation Quantization modules (DQ-modules) on smooth complex (algebraic and analytic) Poisson varieties, in the spirit of the Kyoto school, that is, with the tools of algebraic microlocal analysis. Besides this study, I also recently started to apply these techniques to some problems of Topological data analysis (TDA).

DQ-modules on smooth complex Poisson varieties have been introduced by Maxim Kontsevich in [Kon01]. They can be considered as an extension of the theory of microdifferential modules to arbitrary smooth Poisson varieties. Hence, their study can be approached via the methods of algebraic analysis as done in the foundational work of Masaki Kashiwara and Pierre Schapira [KS12] where a systematic investigation of finiteness, duality and holonomicity properties was carried out.

One of the main motivations of my work on DQ-modules is to understand in which extent derived categories of DQ-modules can be considered as invariants of algebraic Poisson varieties. This led to study the structure of such categories as well as to describe the functors between them. In [GP20], with David Gepner, we identified the linear structure inherent to ∞-categories of DQ-modules. This allowed us to obtain several results. We proved that the ∞-category of qcc modules (the analogue of quasi-coherent sheaves in the theory of DQ-modules) is a deformation of the ∞-category of quasi-coherent sheaves. The characterization of the linear structure of categories of DQ-modules led us to set up the study of integral representations for DQ-modules in the framework of the Morita theory of linear ∞-categories. The approach to integral representations via the Morita theory of linear ∞-categories was initiated by Bertrand Toën in [Toë07] and further developed by David Ben-Zvi, John Francis and David Nadler in [BZFN10]. This setting provides a clear strategy to establish integral representation. Relying on the results of [Pet12, Pet14], we obtained several integral representation theorems for functors between ∞-categories of DQ-modules. In particular, equivalences between ∞-categories of coherent DQ-modules are given by Fourier-Mukai transforms. This, in turns, implies that a linear equivalence between ∞-categories of coherent DQ-modules induces an equivalence between categories of coherent sheaves. These results are presented in Chapter 1.

In Chapter 2, I report on my results concerning the quantization of spectral curves.

In [Pet18c], I showed that under mild conditions the section defining the spectral curve associated to a Higgs bundle can be quantized by a microlocal quantum curve. This implies the existence of a DQ-module quantizing the spectral curve. I also presented several examples in which quantum curves appear as sections of certain algebras of operators to which a DQ-algebra is naturally associated.

Chapter 3 deals with the codimension-three conjecture. This conjecture was formulated for microdifferential modules by Masaki Kashiwara at the end of the seventies. It provides sufficient conditions to extend holonomic microdifferential modules through an analytic subset of their support. In [Pet18a], I proved an analogue result for holonomic DQ-modules. This result cannot be deduced from [KV14] though it borrows several essential techniques. Then, in [Pet18b], by studying F-equivariant DQ-modules, I established an equivalence of categories between coherent DQ-modules on the cotangent bundle endowed with an F-action and coherent microdifferential modules on the projective cotangent bundle. This allowed one to deduce the codimension-three conjecture for formal microdifferential modules from the one for DQ-modules.

As previously mentioned, I recently started to work on the sheaf theoretical approaches to topological data analysis and in particular on the metric aspects of persistent homology. There are several possible formalizations of the notion of (multi)-persistence modules. Moreover, the various categories of persistence modules are equipped with pseudo-distances and are equivalent to categories of sheaves on vector spaces (not necessarily for the usual topology). This raises two questions. Is it possible to extend these pseudo-distances for sheaves on vector spaces to sheaves on metric spaces? In, the case of sheaves on vector spaces, how do these pseudo-distances compare? In the first part of Chapter 4, I summarize some of the results obtained with Pierre Schapira in [PS20]. In this article, we extended the pseudo-distance initially introduced to study persistence modules to sheaves on a large class of metric spaces. This class contains in particular compact Riemannian manifolds. We also showed that several classical integral transforms (e.g. the Fourier-Sato transform) are isometries for these distances. Then, in the second part of this chapter, I sharpen some of the results obtained together with Nicolas Berkouk in [BP18] where we proved that the various formulation of persistence modules in terms of sheaves are isometric.

List of presented works for the habilitation

[PS20] F. Petit and P. Schapira, Thickening of the diagonal, interleaving distance and Fourier-Sato transform, *ArXiv e-print*, arXiv:2006.13150, 2020.

[GP20] D. Gepner and F. Petit, An integral representation theorem for DQ-modules,*ArXiv e-print*, arXiv:2004.10176, 2020

[BP18] N. Berkouk and F. Petit, Ephemeral persistence modules and distance comparison, à paraître dans *Algebraic and Geometric Topology*, *ArXiv e-print*, arXiv:1902.09933, 2019.

[Pet18b] F. Petit, Holomorphic Frobenius actions for DQ-modules, accepté à *Publications of the Research Institute for Mathematical Sciences ArXiv e-print*, arXiv:1803.07923, 2018.

[Pet18c] F. Petit, Quantization of spectral curves and DQ-modules, *Journal of Noncommutative Geometry*, Volume 13, Issue 1, pp. 161-191, 2019.

[Pet18a] F. Petit, The codimension-three conjecture for holonomic DQ-modules, *Selecta Mathematica New Ser*, Volume 24, Issue 2, pp 1147-1181, 2018.

Chapter 1

Integral representation theorem for DQ-modules

1.1 Introduction

A key result in the study of algebraic varieties through their derived categories of coherent sheaves is the theorem of Orlov [Orl97] which states that any exact fully faithful functor between derived categories of coherent sheaves on smooth projective varieties is given by an integral transform such that the kernel is an object of the derived category of coherent sheaves of the product variety. Here, we present an analogue of this result in the framework of DQ-modules (see Theorem 1.44) for functors between ∞-categories of coherent DQ-modules which have a right adjoint.

In [Toë07], Toën developed the Morita theory of dg-categories and set the study of integral representation theorem for quasi-coherent sheaves in this setting. In [BZFN10, BZNP17], the authors furthered the approach of Toën and formulated the problem in the framework of linear ∞-categories. Here, we follow their general strategy but the difficulties are twofold. First, we need to elucidate the linear structure of ∞-categories of DQ-modules as they are not only enriched in $\mathbb{C}[[\hbar]]$-modules but in cohomologically complete $\mathbb{C}[[\hbar]]$-modules. Doing this allows us to prove that the ∞-category of qcc modules (the DQ-modules analogue of quasi-coherent sheaves) with its appropriate linear structure is a deformation of the ∞-category of quasi-coherent sheaves. Second, contrary to the case of the category of quasi-coherent sheaves on a smooth complex algebraic variety, the coherent DQ-modules are not the compact objects of the category of the qcc modules even though the underlying variety is smooth. This difficulty is specific to DQ-modules and the techniques developed to study integral representation theorem for coherent modules on non-smooth spaces (see [BZNP17]) do not seem to apply to the setting of DQ-modules. One of the key ingredients to handle the coherent case is Theorem 1.40 which states that an integral transform preserves the compact objects of the qcc modules if and only if its kernel is coherent.

1.2 DQ-algebras and DQ-modules

1.2.1 DQ-algebras

Our presentation follows [KS12].

We denote by $\mathbb{C}^\hbar := \mathbb{C}[[\hbar]]$ the ring of formal power series with coefficients in $\mathbb{C}$ and by $\mathbb{C}^{\hbar,\mathrm{loc}} := \mathbb{C}((\hbar))$ the field of Laurent series with complex coefficients. Here $(X, \mathcal{O}_X)$ is either a smooth complex algebraic variety or a complex manifold. We write $\mathcal{D}_X$ for the sheaf of holomorphic/algebraic differential operators on X. We define the following sheaf of $\mathbb{C}^\hbar$-algebras

$$\mathcal{O}_X^\hbar := \varprojlim_{n \in \mathbb{N}} \mathcal{O}_X \underset{\mathbb{C}}{\otimes} (\mathbb{C}^\hbar / \hbar^n \mathbb{C}^\hbar).$$

We define similarly the sheaf $\mathcal{D}_X^\hbar$.

Definition 1.1. A star-product denoted $\star$ on $\mathcal{O}_X^\hbar$ is a $\mathbb{C}^\hbar$-bilinear associative multiplicative law satisfying

$$f \star g = \sum_{i \geq 0} P_i(f, g)\hbar^i \text{ for every } f,\ g \in \mathcal{O}_X,$$

where the P_i are bi-differential operators such that for every $f, g \in \mathcal{O}_X, P_0(f, g) = fg$ and $P_i(1, f) = P_i(f, 1) = 0$ for $i > 0$. The pair $(\mathcal{O}_X^\hbar, \star)$ is called a star-algebra.

Given a star algebra $(\mathcal{O}_X^\hbar, \star)$, the quotient $\mathcal{O}_X/\hbar\mathcal{O}_X$ is isomorphic to $\mathcal{O}_X$ as a $\mathbb{C}$-algebra and the isomorphism is unique. Hence, we get a map

$$\sigma_0 \colon \mathcal{O}_X^\hbar \to \mathcal{O}_X/\hbar\mathcal{O}_X \to \mathcal{O}_X.$$

This map comes with a section $\phi \colon \mathcal{O}_X \to \mathcal{O}_X^\hbar$ which identifies functions with degree zero power series. Omitting ϕ, the star-product on $\mathcal{O}_X^\hbar$ induces a Poisson bracket through the following formula

$$\{f; g\} = \frac{f \star g - g \star f}{\hbar} \mod \hbar \quad \text{for } f, g \in \mathcal{O}_X.$$

In the complex setting, star-algebras are too rigid and only allows to quantize a few varieties. From the standpoint of complex geometry, a better suited notion is the one of DQ-algebras.

Definition 1.2. A DQ-algebra $\mathcal{A}_X$ on X is a $\mathbb{C}^\hbar$-algebra locally isomorphic to a star-algebra as a $\mathbb{C}^\hbar$-algebra.

It follows from [KS12, Lemma 2.1.15] that there is a unique isomorphism of $\mathbb{C}$-algebras

$\mathcal{A}_X/\hbar\mathcal{A}_X \xrightarrow{\sim} \mathcal{O}_X$. We write $\sigma_0 : \mathcal{A}_X \twoheadrightarrow \mathcal{O}_X$ for the composition

$$\mathcal{A}_X \twoheadrightarrow \mathcal{A}_X/\hbar\mathcal{A}_X \xrightarrow{\sim} \mathcal{O}_X.$$

Even though, in general the map σ_0 does not admit a section, it is an epimorphism of sheaves and still induces a Poisson bracket $\{\cdot,\cdot\}$ on $\mathcal{O}_X$ defined as follows:

$$\text{for every } a,\ b \in \mathcal{A}_X,\ \{\sigma_0(a), \sigma_0(b)\} = \sigma_0(\hbar^{-1}(ab - ba)).$$

One verifies immediately that a DQ-algebra enjoys the following properties:

$$\begin{cases} \text{(i) } \hbar \text{ is central in } \mathcal{A}_X, \\ \text{(ii) } \mathcal{A}_X \text{ has no } \hbar\text{-torsion (i.e. the map } \mathcal{A}_X \xrightarrow{\hbar} \mathcal{A}_X \text{ is injective)}, \\ \text{(iii) } \mathcal{A}_X \text{ is } \hbar\text{-complete}, \\ \text{(iv) } \mathcal{A}_X/\hbar\mathcal{A}_X \simeq \mathcal{O}_X \text{ as sheaves of } \mathbb{C}\text{-algebras.} \end{cases} \tag{1.2.1}$$

Remark 1.3. A sheaf $\mathcal{A}_X$ of $\mathbb{C}^\hbar$-algebras satisfying condition (1.2.1) is called a formal deformation of $\mathcal{O}_X$. If X is a smooth algebraic variety, then a formal deformation $\mathcal{A}_X$ is equivalent to a DQ-algebra (see for instance [Yek13, Theorem 0.1]). Note that it is not known if such a result holds in the analytic setting. Without this result it is not clear how to define an external product for formal deformation. That is why we do not use condition (1.2.1) as a definition of DQ-algebras. Nonetheless, these conditions are sufficient to develop a large part of the theory of DQ-modules and DQ-algebras as it is visible from [KS12, Chapter 1].

We recall the following consequence of [KS12, Theorem 1.2.5]

Theorem 1.4. *Let $\mathcal{A}_X$ be a DQ-algebra. Then $\mathcal{A}_X$ is a Noetherian sheaf.*

The notion of DQ-algebra is yet to rigid to ensure that it is always possible to quantize by deformation a smooth complex (algebraic or analytic) variety. One need the notion of DQ-algebroid as the gluing datum are also deformed in the quantization process and the object obtained are not anymore sheaves.

Definition 1.5. A deformation quantization algebroid stack (or DQ-algebroid for short) on a smooth complex variety $(X, \mathcal{O}_X)$ is a stack in $\mathbb{C}^\hbar$-enriched categories, locally non-empty, locally connected and such that for every open set $U \in X$ and $\sigma \in \mathcal{A}_X(U)$, the $\mathbb{C}^\hbar_X$ algebra $\mathcal{E}\mathrm{nd}_{\mathcal{A}_X}(\sigma)$ is a DQ-algebra on U.

We list below a few important properties of DQ-algebroid and refer the reader to [KS12, §2.2] for more details

- If $\mathcal{A}_X$ is a DQ-algebroid then $\mathcal{A}_X^{\mathrm{op}}$ is again a DQ-algebroid and we denote it $\mathcal{A}_{X^a}$.

- Let X and Y be two (algebraic or analytic) smooth varieties endowed with DQ-algebroid $\mathcal{A}_X$ and $\mathcal{A}_Y$. Then $X \times Y$ is endowed with a DQ-algebroid $\mathcal{A}_X \boxtimes \mathcal{A}_Y$. It contains $\mathcal{A}_X \underset{\mathbb{C}^\hbar}{\boxtimes} \mathcal{A}_Y$ as a $\mathbb{C}^\hbar$-algebroid.

1.2.2 DQ-modules

∞-Derived category of DQ-modules

Let $(X, \mathcal{O}_X)$ be a smooth complex algebraic (or analytic) variety endowed with a DQ-algebroid $\mathcal{A}_X$. We denote by $\mathrm{Mod}(\mathcal{A}_X)$ the Grothendieck abelian category of left $\mathcal{A}_X$-modules and its category of chain complexes $\mathrm{Ch}(\mathcal{A}_X)$ can be endowed with the injective model structure where the quasi-equivalences are the quasi-isomorphism and the cofibrations are the monomorphism. The ∞-category presented by $\mathrm{Ch}(\mathcal{A}_X)$ endowed with this model structure is denoted $\mathscr{D}(\mathcal{A}_X)$. Its homotopy category is equivalent to $\mathrm{D}(\mathcal{A}_X)$ the usual derived category of $\mathrm{Mod}(\mathcal{A}_X)$. We write $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{A}_X)$ for the full abelian subcategory of $\mathrm{Mod}(\mathcal{A}_X)$ the object of which are the coherent DQ-modules and $\mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X)$ for the full subcategory of $\mathscr{D}(\mathcal{A}_X)$ whose objects are the bounded complex of DQ-modules with coherent cohomology. We write $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X)$ for its homotopy category. We use similar notations for $\mathcal{A}_X^{\mathrm{loc}}$-modules. The ∞-category $\mathscr{D}(\mathcal{A}_X)$ is a stable presentable $\mathbb{C}^\hbar$-linear category. This implies that $\mathscr{D}(\mathcal{A}_X)$ has a $\mathscr{D}(\mathbb{C}^\hbar)$-enriched mapping space functor

$$\underline{\mathrm{Map}}_{\mathcal{A}_X}(-\,,-)\colon \mathscr{D}(\mathcal{A}_X)^{\mathrm{op}} \times \mathscr{D}(\mathcal{A}_X) \to \mathscr{D}(\mathbb{C}^\hbar).$$

Moreover, the functor $\mathscr{H}om_{\mathcal{A}_X}(-,-)\colon \mathrm{Mod}(\mathcal{A}_X)^{\mathrm{op}} \times \mathrm{Mod}(\mathcal{A}_X) \to \mathrm{Mod}(\mathbb{C}^\hbar_X)$ is left exact. Using the injective model structure on $\mathrm{Ch}(\mathcal{A}_X)$, we obtain a right derived functor

$$\mathscr{M}ap_{\mathcal{A}_X}(-,-)\colon \mathscr{D}(\mathcal{A}_X)^{\mathrm{op}} \times \mathscr{D}(\mathcal{A}_X) \to \mathscr{D}(\mathbb{C}^\hbar_X).$$

Note that $\Gamma(X, \mathscr{M}ap_{\mathcal{A}_X}(-,-)) \simeq \underline{\mathrm{Map}}_{\mathcal{A}_X}(-,-)$.

There is a $\mathbb{C}$-algebroid denoted $\mathrm{gr}_\hbar \mathcal{A}_X$ associated with the DQ-algebroid $\mathcal{A}_X$. The algebroid $\mathrm{gr}_\hbar \mathcal{A}_X$ is an invertible $\mathcal{O}_X$-algebroid and comes together with a canonical morphism of $\mathbb{C}$-algebroids

$$\mathcal{A}_X \to \mathrm{gr}_\hbar \mathcal{A}_X.$$

This algebroid induces a $\mathrm{gr}_\hbar \mathcal{A}_X \otimes_{\mathbb{C}^\hbar_X} \mathcal{A}_{X^{\mathrm{a}}}$-module still denoted $\mathrm{gr}_\hbar \mathcal{A}_X$. As an $\mathcal{A}_X \otimes \mathcal{A}_{X^{\mathrm{a}}}$-module, $\mathrm{gr}_\hbar \mathcal{A}_X$ is isomorphic to $\mathbb{C} \otimes_{\mathbb{C}^\hbar} \mathcal{A}_X$. This bi-module induces a functor

$$\mathrm{gr}_\hbar\colon \mathrm{Mod}(\mathcal{A}_X) \to \mathrm{Mod}(\mathrm{gr}_\hbar \mathcal{A}_X), \quad \mathcal{M} \mapsto \mathrm{gr}_\hbar \mathcal{A}_X \otimes_{\mathcal{A}_X} \mathcal{M}. \tag{1.2.2}$$

Moreover, the functor $\mathrm{gr}_\hbar$ is left adjoint to the exact functor $\iota\colon \mathrm{Mod}(\mathrm{gr}_\hbar \mathcal{A}_X) \to \mathrm{Mod}(\mathcal{A}_X)$ induced by the canonical morphism $\mathcal{A}_X \to \mathrm{gr}_\hbar \mathcal{A}_X$. On an algebraic variety, the algebroid stack $\mathrm{gr}_\hbar \mathcal{A}_X$ is equivalent to the stackification of the structure sheaf $\mathcal{O}_X$ (see [KS12, Remark 2.1.17]). This induces an equivalence between $\mathrm{Mod}(\mathcal{O}_X)$ and $\mathrm{Mod}(\mathrm{gr}_\hbar \mathcal{A}_X)$. Using the semi-free model structure introduced in [GP20, Appendix B.1], one shows that the functor (1.2.2) induces a morphism in $\mathrm{Mod}_{\mathscr{D}(\mathbb{C}^\hbar)}$ informally defined by

$$\mathrm{gr}_\hbar \colon \mathscr{D}(\mathcal{A}_X) \to \mathscr{D}(\mathrm{gr}_\hbar \mathcal{A}_X), \quad \mathcal{M} \mapsto \mathrm{gr}_\hbar \mathcal{A}_X \overset{\mathrm{L}}{\underset{\mathcal{A}_X}{\otimes}} \mathcal{M}. \tag{1.2.3}$$

1.2.3 Cohomologically complete modules

Here, due to its influence on our work, we give a brief account of the notion of cohomologically complete module as it initially appears in [KS12] and explain how it naturally lead to consider enrichment in cohomologically complete modules for categories of DQ-modules.

In [KS12], the authors establish several fundamental finiteness and duality results for DQ-modules. In particular, they prove an analogue of Grauert's finiteness theorem for coherent DQ-modules by reducing the non-commutative version to the classical one. To that end, they introduce the notion of cohomological completeness, a concept closely related to derived adic-completion (see [Lur18] and [PSY12] for a comparison of the two notions), and defined in term of semi-orthogonal decomposition of the triangulated categories of DQ-modules. The notion of cohomological completion was vastly generalized by several authors in various context, culminating in the treatment of J. Lurie in [Lur11b, Lur18], and plays a key role in our work (see [Pet14, GP20]).

We consider the functors.

$$\mathrm{gr}_\hbar \colon \mathrm{D}(\mathcal{A}_X) \to \mathrm{D}(\mathcal{O}_X), \quad \mathcal{M} \mapsto \mathcal{O}_X \overset{\mathrm{L}}{\underset{\mathcal{A}_X}{\otimes}} \mathcal{M}, \tag{1.2.4}$$

$$(\cdot)^{\mathrm{loc}} \colon \mathrm{D}(\mathcal{A}_X) \to \mathrm{D}(\mathcal{A}_X^{\mathrm{loc}}), \quad \mathcal{M} \mapsto \mathcal{A}_X^{\mathrm{loc}} \overset{\mathrm{L}}{\underset{\mathcal{A}_X}{\otimes}} \mathcal{M}. \tag{1.2.5}$$

The functor $(\cdot)^{\mathrm{loc}}$ has a fully faithful right adjoint. Hence, the category $\mathrm{D}(\mathcal{A}_X^{\mathrm{loc}})$ is equivalent to the full subcategory of $\mathrm{D}(\mathcal{A}_X)$ spanned by the objects $\mathcal{M}$ satisfying one of the following equivalent conditions:

(a) $\mathrm{gr}_\hbar \mathcal{M} \simeq 0$,

(b) the morphism $\mathcal{M} \to \mathcal{A}_X^{\mathrm{loc}} \underset{\mathcal{A}_X}{\otimes} \mathcal{M}$ induced by the canonical localization map $\mathcal{A}_X \to \mathcal{A}_X^{\mathrm{loc}}$ is an isomorphism.

Condition (a) suggests to consider the right orthogonal $\mathrm{D}(\mathcal{A}_X^{\mathrm{loc}})^{\perp r}$ of $\mathrm{D}(\mathcal{A}_X^{\mathrm{loc}})$ inside $\mathrm{D}(\mathcal{A}_X)$. More generally, Let $\mathcal{R}$ be a $\mathbb{Z}[\hbar]_X$-algebra without $\hbar$-torsion and set $\mathcal{R}^{\mathrm{loc}} = \mathbb{Z}[\hbar, \hbar^{-1}] \otimes_{\mathbb{Z}[\hbar]} \mathcal{R}$.

Definition 1.6. We say that an object $\mathcal{M}$ of $\mathrm{D}(\mathcal{R})$ is cohomologically complete if $\mathcal{M} \in \mathrm{D}(\mathcal{R}^{\mathrm{loc}})^{\perp r}$. We write $\mathrm{D}_{\mathrm{cc}}(\mathcal{R})$ for $\mathrm{D}(\mathcal{R}^{\mathrm{loc}})^{\perp r}$.

In particular one shows that $\mathcal{M}$ is cohomologically complete if and only if

$$\mathrm{R}\mathcal{H}om_{\mathcal{R}}(\mathcal{R}^{\mathrm{loc}}, \mathcal{M}) \simeq 0.$$

Cohomologically complete modules have the following properties.

Proposition 1.7 ([KS12, Corollary 1.4.6]). *The restriction of the functor* $\mathrm{gr}_\hbar$ *to* $\mathrm{D}_{\mathrm{cc}}(\mathcal{R})$ *is conservative.*

Proposition 1.8 ([KS12, Proposition 1.5.10 and 1.5.12]). *(i) Let* $\mathcal{M} \in \mathrm{D}_{\mathrm{cc}}(\mathcal{R})$*, then for every* $\mathcal{N} \in \mathrm{D}(\mathcal{R})$*,* $\mathrm{R}\mathcal{H}om_{\mathcal{R}}(\mathcal{N}, \mathcal{M}) \in \mathrm{D}_{\mathrm{cc}}(\mathbb{Z}[\hbar_X])$

(ii) Let $f : X \to Y$ *a continuous map. Let* $\mathcal{M} \in \mathrm{D}_{\mathrm{cc}}(\mathbb{Z}[\hbar]_X)$*. Then* $\mathrm{R}f_*\mathcal{M}$ *is cohomologically complete.*

The preceding proposition implies that if $\mathcal{M} \in \mathrm{D}_{\mathrm{cc}}(\mathcal{A}_X)$, then for every $\mathcal{N} \in \mathrm{D}(\mathcal{A}_X)$, $\mathrm{RHom}_{\mathcal{A}_X}(\mathcal{N}, \mathcal{M}) \in \mathrm{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)$. This suggest that ∞-categories of DQ-modules should be enriched in cohomologically complete modules. In the next section, we explain how to handle ∞-categories enriched in cohomologically complete modules by using Lurie's notion of linear ∞-categories.

1.3 $\hbar$-complete ∞-categories

In this section, we review the necessary notions to handle ∞-categories enriched in cohomologically complete $\mathbb{C}^\hbar$-modules. Our approach relies mainly on [Lur18, Chapter 7].

We denote by $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ the ∞-category of presentable stable ∞-categories with morphisms the colimit preserving functors. If $\mathscr{C}$ and $\mathscr{C}'$ are two objects of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, we write $\mathrm{Fun}^{\mathrm{L}}(\mathscr{C}, \mathscr{C}')$ for the functor from $\mathscr{C}$ to $\mathscr{C}'$ commuting with colimits (or equivalently having a right adjoint). If $\mathscr{C}$ is an object of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, we write $\mathscr{C}^\omega$ for the ∞-subcategory spanned by its compact object. It is an object of $\mathrm{Cat}^{\mathrm{ex}}_{\infty,\mathrm{idem}}$ the ∞-category of idempotent complete stable ∞-categories with exact functors as morphisms.

We write $\mathscr{D}(\mathbb{C}^\hbar)$ for the ∞-derived category of $\mathbb{C}^\hbar$-modules. It can naturally be endowed with a structure presentably symmetric monoidal ∞-category ()we will still write $\mathscr{D}(\mathbb{C}^\hbar)$ when endowing this category with its canonical monoidal structure). We denote by $\mathrm{Mod}_{\mathscr{D}(\mathbb{C}^\hbar)}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ the ∞-category of left $\mathscr{D}(\mathbb{C}^\hbar)$-modules in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. For brevity, we will often write $\mathrm{Mod}_{\mathscr{D}(\mathbb{C}^\hbar)}$. A $\mathbb{C}^\hbar$-linear ∞-category is an object of $\mathrm{Mod}_{\mathscr{D}(\mathbb{C}^\hbar)}$ and a $\mathbb{C}^\hbar$-linear functor is a morphism of left $\mathscr{D}(\mathbb{C}^\hbar)$-modules in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. If $\mathscr{M}$ and $\mathscr{N}$ are two $\mathscr{D}(\mathbb{C}^\hbar)$-modules we denote by $\mathrm{Fun}^{\mathrm{L}}_{\mathscr{D}(\mathbb{C}^\hbar)}(\mathscr{M}, \mathscr{N})$ the ∞-category of $\mathbb{C}^\hbar$-linear functors and write $\mathrm{Fun}^{\mathrm{L}}_{\mathscr{D}(\mathbb{C}^\hbar),\omega}(\mathscr{M}, \mathscr{N})$ for its full subcategory spanned by the morphisms preserving the compact objects.

If $\mathscr{M}$ is a $\mathbb{C}^\hbar$-linear category then $\mathscr{M}^\omega \in \mathrm{Mod}_{\mathscr{D}^\omega(\mathbb{C}^\hbar)}(\mathrm{Cat}^{\mathrm{ex}}_{\infty,\mathrm{idem}})$ similarly the category $\mathrm{Fun}^{\mathrm{L}}_{\mathscr{D}(\mathbb{C}^\hbar),\omega}(\mathscr{M}, \mathscr{N})$ is a $\mathscr{D}^\omega(\mathbb{C}^\hbar)$-module.

The notion of $\mathbb{C}^\hbar$-linear categories provides a convenient way of handling presentable stable categories enriched in $\mathscr{D}(\mathbb{C}^\hbar)$.

Let $\mathscr{C}$ be a $\mathbb{C}^\hbar$-linear category. The enriched mapping space functor of $\mathscr{C}$ is denoted

$$\underline{\mathrm{Map}}_{\mathscr{C}} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{D}(\mathbb{C}^\hbar).$$

We refer the reader to [Lur18, Appendix D.7.1] for more details.

Given $\mathscr{C} \in \mathrm{Mod}_{\mathscr{D}(\mathbb{C}^\hbar)}$, we say, following Lurie, that

Definition 1.9. (i) An object $N \in \mathscr{C}$ is $\hbar$-nilpotent if the localization $\mathbb{C}^{\hbar,\mathrm{loc}} \otimes_{\mathbb{C}^\hbar} N$ vanishes.

(ii) An object $L \in \mathscr{C}$ is $\hbar$-local if the canonical map $L \to \mathbb{C}^{\hbar,\mathrm{loc}} \otimes_{\mathbb{C}^\hbar} L$ is an equivalence.

(iii) An object $M \in \mathscr{C}$ is $\hbar$-cohomologically complete if $M^{\mathbb{C}^{\hbar,\mathrm{loc}}} \simeq 0$.

We denote by $\mathscr{C}^{\mathrm{nil}}$, $\mathscr{C}^{\mathrm{loc}}$, $\mathscr{C}^{\mathrm{cc}}$ the full subcategories of $\mathscr{C}$ respectively spanned by the nilpotent, local and complete objects. We write

$$i_\vee \colon \mathscr{C}^{\mathrm{nil}} \subset \mathscr{C} \qquad j_* \colon \mathscr{C}^{\mathrm{loc}} \subset \mathscr{C} \qquad i_\wedge \colon \mathscr{C}^{\mathrm{cc}} \subset \mathscr{C}$$

for the respective fully faithful inclusion, $i^\vee$ for the right adjoint of $i_\vee$, and j^* and $i^\wedge$ for the left adjoints of j_* and $i_\wedge$, respectively.

Remark 1.10. Let $\mathcal{A}_X$ be a DQ-algebroid. An object of $\mathscr{D}(\mathcal{A}_X)$ is cohomologically complete in the sense of Definition 1.6 if and only if it is cohomologically complete in the sense of Definition 1.9.

The following result is [Lur18, Proposition 7.3.1.4]

Proposition 1.11. *Any $\mathscr{D}(\mathbb{C}^\hbar)$-linear ∞-category $\mathscr{M}$ admits a semiorthogonal decomposition of the form $(\mathscr{M}^{\mathrm{loc}}, \mathscr{M}^{\mathrm{cc}})$.*

Consider the category $\mathscr{D}(\mathbb{C}^\hbar)$. We write respectively $\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)$ and $\mathscr{D}_{\mathrm{loc}}(\mathbb{C}^\hbar)$ for the full subcategories of cohomologically complete and $\hbar$-local objects. In particular, $\mathscr{D}_{\mathrm{loc}}(\mathbb{C}^\hbar)$ and $\mathscr{D}(\mathbb{C}^{\hbar,\mathrm{loc}})$ are equivalent as $\mathbb{C}^\hbar$-linear categories. We have the following result regarding $\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)$ which is a consequence of [Lur18, Variant 7.3.5.6] and [GP20, Corollary A.16]

Proposition 1.12. *The ∞-category $\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)$ admits the structure of an idempotent presentably symmetric monoidal ∞-category under $\mathscr{D}(\mathbb{C}^\hbar)$. That is, there exists a presentably symmetric monoidal structure on $\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)$ such that the functor $i^\wedge : \mathscr{D}(\mathbb{C}^\hbar) \to \mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)$ is symmetric monoidal and the tensor product functor $\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar) \times \mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar) \to \mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)$ is given on objects by the formula $(M, N) \mapsto i^\wedge (i_\wedge(M) \otimes_{\mathbb{C}^\hbar} i_\wedge(N))$.*

The symmetric monoidal functor $i^\wedge : \mathscr{D}(\mathbb{C}^\hbar) \to \mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)$ induces a forgetful functor

$$i_*\colon \operatorname{Mod}_{\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)} \to \operatorname{Mod}_{\mathscr{D}(\mathbb{C}^\hbar)}$$

which admits a left adjoint functor

$$i^* : \operatorname{Mod}_{\mathscr{D}(\mathbb{C}^\hbar)} \to \operatorname{Mod}_{\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)}, \quad \mathcal{M} \mapsto \mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar) \underset{\mathscr{D}(\mathbb{C}^\hbar)}{\otimes} \mathcal{M}.$$

Since $\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)$ is idempotent under $\mathscr{D}(\mathbb{C}^\hbar)$ this implies that $i_*\colon \operatorname{Mod}_{\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)} \to \operatorname{Mod}_{\mathscr{D}(\mathbb{C}^\hbar)}$ is fully faithful. The functor $i^* : \operatorname{Mod}_{\mathscr{D}(\mathbb{C}^\hbar)} \to \operatorname{Mod}_{\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)}$ is the functor of cohomological completion.

The following proposition and corollary allow to compute cohomological completion of categories and functors.

Proposition 1.13 ([GP20, Proposition 2.9])**.** *For any $\mathbb{C}^\hbar$-linear ∞-category $\mathscr{M}$, there is a commutative diagram of Verdier sequences*

$$\begin{array}{ccccc}
\mathscr{D}_{\mathrm{loc}}(\mathbb{C}^\hbar) \underset{\mathscr{D}(\mathbb{C}^\hbar)}{\otimes} \mathscr{M} & \longrightarrow & \mathscr{D}(\mathbb{C}^\hbar) \underset{\mathscr{D}(\mathbb{C}^\hbar)}{\otimes} \mathscr{M} & \longrightarrow & \mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar) \underset{\mathscr{D}(\mathbb{C}^\hbar)}{\otimes} \mathscr{M} \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{M}^{\mathrm{loc}} & \longrightarrow & \mathscr{M} & \longrightarrow & \mathscr{M}^{\mathrm{cc}}
\end{array}$$

in which the vertical maps are equivalences.

As a corollary, we get

Corollary 1.14 ([GP20, Corollary 2.11])**.** *Let $f^*\colon \mathscr{M} \to \mathscr{N}$ be a morphism in* $\operatorname{Mod}_{\mathscr{D}(\mathbb{C}^\hbar)}$*. The diagram*

$$\begin{array}{ccc}
\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar) \otimes_{\mathscr{D}(\mathbb{C}^\hbar)} \mathscr{M} & \xrightarrow{\mathrm{id}_{\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)} \otimes f^*} & \mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar) \otimes_{\mathscr{D}(\mathbb{C}^\hbar)} \mathscr{N} \\
\downarrow & & \downarrow \\
\mathscr{M}^{\mathrm{cc}} & \xrightarrow{i^\wedge \circ f^* \circ i_\wedge} & \mathscr{N}^{\mathrm{cc}},
\end{array}$$

where the vertical maps are the equivalences of Proposition 1.13, commutes up to natural equivalence. Moreover, the composite $i^\wedge \circ f^* \circ i_\wedge : \mathscr{M}^{\mathrm{cc}} \to \mathbb{N}^{\mathrm{cc}}$ is $\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)$-linear.

In practice, we are mainly dealing with categories for which all the objects are either $\hbar$-nilpotent or $\hbar$-local or cohomologically complete.

Definition 1.15. (i) A stable $\mathbb{C}^\hbar$-linear ∞-category $\mathscr{C}$ is $\hbar$-nilpotent if every object of $\mathscr{C}$ is $\hbar$-nilpotent.

(ii) A stable $\mathbb{C}^\hbar$-linear ∞-category $\mathscr{C}$ is $\hbar$-local if every object of $\mathscr{C}$ is $\hbar$-local.

(iii) A stable $\mathbb{C}^\hbar$-linear ∞-category $\mathscr{C}$ is cohomologically complete if every object of $\mathscr{C}$ is cohomologically complete.

We now give several characterizations of cohomologically complete categories.

Proposition 1.16 ([GP20, Proposition 1.14])**.** *Let $\mathscr{C}$ be a stable $\mathbb{C}^{\hbar}$-linear ∞-category. The following statements are equivalent:*

(i) $\mathscr{C}$ is cohomologically complete.

(ii) For every M and N in $\mathscr{C}$, $\underline{\mathrm{Map}}(M,N) \in \mathscr{D}_{\mathrm{cc}}(\mathbb{C}^{\hbar})$.

(iii) $\mathscr{C}$ is $\hbar$-nilpotent.

(iv) The action $\mathscr{D}(\mathbb{C}^{\hbar}) \otimes \mathscr{C} \to \mathscr{C}$ factors through the map $i^{\wedge} \otimes \mathrm{id} : \mathscr{D}(\mathbb{C}^{\hbar}) \otimes \mathscr{C} \to \mathscr{D}_{\mathrm{cc}}(\mathbb{C}^{\hbar}) \otimes \mathscr{C}$.

(v) $\mathscr{C} \in \mathrm{Mod}_{\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^{\hbar})}$.

Example 1.17. Recall that the category $\mathscr{D}(\mathcal{A}_X)$ is an object of $\mathrm{Mod}_{\mathscr{D}(\mathbb{C}^{\hbar})}$. Moreover, the category $\mathscr{D}(\mathrm{gr}_{\hbar}\, \mathcal{A}_X)$ is a $\mathbb{C}$-linear category. Hence, it is $\mathbb{C}^{\hbar}$-linear via the presentably symmetric monoidal functor

$$\mathrm{gr}_{\hbar} \colon \mathscr{D}(\mathbb{C}^{\hbar}) \to \mathscr{D}(\mathbb{C}). \tag{1.3.1}$$

This implies in particular that $\mathscr{D}(\mathrm{gr}_{\hbar}\, \mathcal{A}_X)$ is cohomologically complete (by Proposition 1.16 (iii)) and thus equivalent to its cohomological completion i.e. $i_* i^*(\mathscr{D}(\mathrm{gr}_{\hbar}\, \mathcal{A}_X)) \simeq \mathscr{D}(\mathrm{gr}_{\hbar}\, \mathcal{A}_X)$.

1.4 The ∞-categories of qcc and coherent DQ-modules

In this subsection, we introduce the ∞-versions of several categories of DQ-modules together with their $\mathbb{C}^{\hbar}$-linear structures.

1.4.1 The ∞-category of qcc-modules

In [Pet12], we introduced the triangulated category $\mathrm{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ of qcc modules and claimed that this category should be thought as a deformation of $\mathrm{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ while deforming $\mathcal{O}_X$ into $\mathcal{A}_X$. Here, we define the ∞-category of qcc-modules together with its $\mathbb{C}^{\hbar}$-linear structure. We use this structure to prove that $\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ is a deformation of $\mathscr{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$.

The categories $\mathscr{D}_{\mathrm{qcoh}}(\mathrm{gr}_{\hbar}\, \mathcal{A}_X)$ is a full $\mathbb{C}$-linear subcategory of $\mathscr{D}(\mathrm{gr}_{\hbar}\, \mathcal{A}_X)$ as well as $\mathbb{C}^{\hbar}$-linear full subcategory via the functor (1.3.1). The ∞-category $\mathscr{D}_{\mathrm{gqcoh}}(\mathcal{A}_X)$ of graded quasicoherent $\mathcal{A}_X$-modules is defined as the pullback

$$\begin{array}{ccc} \mathscr{D}_{\mathrm{gqcoh}}(\mathcal{A}_X) & \longrightarrow & \mathscr{D}(\mathcal{A}_X) \\ \downarrow & & \downarrow {\scriptstyle \mathrm{gr}_{\hbar}} \\ \mathscr{D}_{\mathrm{qcoh}}(\mathrm{gr}_{\hbar}\, \mathcal{A}_X) & \hookrightarrow & \mathscr{D}(\mathrm{gr}_{\hbar}\, \mathcal{A}_X) \end{array} \tag{1.4.1}$$

in $\mathrm{Mod}_{\mathscr{D}(\mathbb{C}^{\hbar})}$. In particular, we view $\mathscr{D}_{\mathrm{gqcoh}}(\mathcal{A}_X)$ as an object of $\mathrm{Mod}_{\mathscr{D}(\mathbb{C}^{\hbar})}$.

In other words, the ∞-category $\mathscr{D}_{\mathrm{gqcoh}}(\mathcal{A}_X)$ is the full $\mathbb{C}^\hbar$-linear sub-category of $\mathscr{D}(\mathcal{A}_X)$ spanned by the objects $\mathcal{M}$ such that $\mathrm{gr}_\hbar \mathcal{M} \in \mathscr{D}_{\mathrm{qcoh}}(\mathrm{gr}_\hbar \mathcal{A}_X)$.

Definition 1.18. The ∞-category $\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ of graded quasicoherent cohomologically complete $\mathcal{A}_X$-modules is the cohomological completion of $\mathscr{D}_{\mathrm{gqcoh}}(\mathcal{A}_X)$, viewed as an object of $\mathrm{Mod}_{\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)}$ i.e. $\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X) := i^*(\mathscr{D}_{\mathrm{gqcoh}}(\mathcal{A}_X))$.

We refer to graded quasicoherent cohomologically complete $\mathcal{A}_X$-modules simply as qcc-modules.

Remark 1.19. It follows from the definition of $\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ that it is a full $\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)$-linear full subcategory of $\mathscr{D}_{\mathrm{cc}}(\mathcal{A}_X)$. Note that $\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ without its $\mathbb{C}^\hbar$-linear structure is the full subcategory of $\mathscr{D}(\mathcal{A}_X)$ spanned by the cohomologically complete graded quasi-coherent objects.

The presentable stable category $\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ is a deformation of the presentable stable ∞-category $\mathscr{D}_{\mathrm{qcoh}}(\mathrm{gr}_\hbar \mathcal{A}_X)$ in the following sense.

Definition 1.20. A formal deformation of a $\mathbb{C}$-linear ∞-category $\mathscr{C}$ is a pair $(\mathscr{C}^\hbar, \mu)$ where $\mathscr{C}^\hbar$ is an object of $\mathrm{Mod}_{\mathscr{D}(\mathbb{C}^\hbar)}$ and μ is a $\mathscr{D}(\mathbb{C})$-linear equivalence

$$\mu\colon \mathscr{D}(\mathbb{C}) \otimes_{\mathscr{D}(\mathbb{C}^\hbar)} \mathscr{C}^\hbar \to \mathscr{C}.$$

We refer the reader to [Lur11a, Section 5.3] for details.

The morphism of commutative algebra $\mathrm{gr}_\hbar\colon \mathscr{D}(\mathbb{C}^\hbar) \to \mathscr{D}(\mathbb{C})$ induces a base change functor

$$\mathrm{Mod}_{\mathscr{D}(\mathbb{C}^\hbar)} \to \mathrm{Mod}_{\mathscr{D}(\mathbb{C})}, \quad \mathscr{C} \mapsto \mathscr{D}(\mathbb{C}) \otimes_{\mathscr{D}(\mathbb{C}^\hbar)} \mathscr{C}$$

which is left adjoint to the forgetful functor $\mathrm{Mod}_{\mathscr{D}(\mathbb{C})} \to \mathrm{Mod}_{\mathscr{D}(\mathbb{C}^\hbar)}$ (that we omit to write). By the construction of $\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X)$, the functor $\mathrm{gr}_\hbar$ induces a $\mathbb{C}^\hbar$-linear functor

$$\mathrm{gr}_\hbar\colon \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X) \to \mathscr{D}_{\mathrm{qcoh}}(\mathrm{gr}_\hbar \mathcal{A}_X).$$

By adjunction, this induces a morphism in $\mathrm{Mod}_{\mathscr{D}(\mathbb{C})}$

$$\Psi\colon \mathscr{D}(\mathbb{C}) \otimes_{\mathscr{D}(\mathbb{C}^\hbar)} \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X) \to \mathscr{D}_{\mathrm{qcoh}}(\mathrm{gr}_\hbar \mathcal{A}_X). \tag{1.4.2}$$

We recall the following result concerning qcc-modules.

Proposition 1.21 ([Pet12, Prop 2.17])**.** *The functors*

$$\mathrm{D}_{\mathrm{qcc}}(\mathcal{A}_X) \underset{\iota}{\overset{\mathrm{gr}_\hbar}{\rightleftarrows}} \mathrm{D}_{\mathrm{qcoh}}(\mathrm{gr}_\hbar \mathcal{A}_X)$$

preserve compact generators

The above result together with the seminal result of [BVdB03] asserting that $\mathscr{D}_{\mathrm{qcoh}}(\mathcal{O}_X)$ is compactly generated by a single generator implies that

Theorem 1.22 ([GP20, Theroem 3.30])**.** *The morphism* (1.4.2) *is an equivalence in* $\mathrm{Mod}_{\mathscr{D}(\mathbb{C})}$.

Corollary 1.23 ([GP20, Corollary 3.31])**.** *The pair* $(\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X), \Psi)$ *is a deformation of* $\mathscr{D}_{\mathrm{qcoh}}(\mathrm{gr}_\hbar \mathcal{A}_X)$.

Remark 1.24. We expect that a stronger result is in fact true. Namely, that the ∞-stack of qcc-modules is a deformation of the stack of quasi-coherent modules. The quantization of zero shifted Poisson structure has only been studied in [Pri18] where the quantization is in term of A_∞-deformation of the dg-category of perfect complexes. We would like to reexamine Pridham work in the light of the linear structure found on cohomologically complete categories.

1.4.2 The ∞-category of coherent DQ-modules

We start by recalling an important property of coherent DQ-modules.

Theorem 1.25 ([KS12, Theorem 1.3.1])**.** *Let* $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X)$*. Then* $\mathcal{M}$ *is cohomologically complete.*

It is clear that if $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X)$ then $\mathrm{gr}_\hbar \mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{gr}_\hbar \mathcal{A}_X)$. This, together with Theorem 1.25, implies that $\mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X)$ is a full ∞-subcategory of $\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X)$. Summarizing, we obtain the following proposition.

Proposition 1.26. *The category* $\mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X)$ *is a full idempotent stable* ∞*-subcategory of* $\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X)$*. Moreover, the* $\mathscr{D}(\mathbb{C}^\hbar)$*-module structure on* $\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ *induces a* $\mathrm{Perf}(\mathbb{C}^\hbar)$*-module structure on* $\mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X)$*; i.e.,* $\mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X) \in \mathrm{Mod}_{\mathrm{Perf}(\mathbb{C}^\hbar)}(\mathrm{Cat}^{\mathrm{ex}}_{\infty,\mathrm{idem}})$.

In view of the definition of the qcc-modules, it is natural to wonder what are the cohomologically complete objects with a coherent associated graded modules. The following result answer the question.

Theorem 1.27 ([KS12, Theorem 1.6.4])**.** *Let* $\mathcal{M} \in \mathrm{D}^{+}(\mathcal{A}_X)$ *and assume that:*

(i) $\mathcal{M}$ *is cohomologically complete,*

(ii) $\mathrm{gr}_\hbar \mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_X)$.

Then $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X)$.

We have the following result regarding coherent modules.

Proposition 1.28 ([KS12, Corollary 2.3.5]). *Let X be a smooth complex analytic (reps. algebraic) variety of dimension d_X. Any coherent $\mathcal{A}_X$-module locally admits a resolution of length at most $d+1$ by free $\mathcal{A}_X$-modules of finite rank.*

It follows from Proposition 1.28 that coherent DQ-modules are perfect. It would be tempting to think they are compact objects of $\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X)$. Nonetheless, this is not the case. Indeed, in [Pet12], we characterized the compact object of $\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X)$. We recall this result

Theorem 1.29 ([Pet12, Theorem 3.20]). *The compact objects of $\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ are the $\mathcal{M}$ such that $\mathcal{M} \in \mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X)$ and $\mathcal{A}_X^{\mathrm{loc}} \underset{\mathcal{A}_X}{\otimes} \mathcal{M} = 0$ (with $\mathcal{M}$ considered as an object of $\mathscr{D}(\mathcal{A}_X)$).*

Remark 1.30. The above theorem implies that in general coherent sheaves are not compact. For instance, $\mathbb{C}^\hbar$ is a coherent $\mathbb{C}^\hbar$-module but is not compact in $\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)$.

The coherent modules are not the compact objects of $\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X)$. This raises the questions of their categorical characterization. Even though $\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ is not a monoidal category, it is still possible to make sense of the notion of dualizable object in $\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X)$.

Definition 1.31. Let $\mathcal{M} \in \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X)$. The module $\mathcal{M}$ is dualizable if the canonical morphism

$$\psi\colon \mathscr{M}\mathrm{ap}_{\mathcal{A}_X}(\mathcal{M}, \mathcal{A}_X) \underset{\mathcal{A}_X}{\otimes} (-) \to \mathscr{M}\mathrm{ap}_{\mathcal{A}_X}(\mathcal{M}, -)$$

is an equivalence of functors $\mathscr{D}(\mathcal{A}_X) \to \mathscr{D}(\mathbb{C}^\hbar_X)$.

We obtain the following characterization of coherent modules.

Proposition 1.32 ([GP20, Proposition 3.24]). *Let $\mathcal{M} \in \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X)$. The module $\mathcal{M}$ is dualizable if and only if $\mathcal{M} \in \mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X)$.*

Hence, for DQ-modules the coherent modules are the dualizable ones while the compact ones are the coherent modules of $\hbar$-torsion.

1.5 Fourier-Mukai transform for DQ-modules

1.5.1 Composition of kernels

In this subsection, we briefly review operations on DQ-modules. They were first introduced in [KS12]. In [Pet14, GP20], we adapted these operations to qcc modules. We use the following notational convention.

Notations 1.33.
1. Consider a product of smooth complex varieties $X_1 \times X_2 \times X_3$, we write it X_{123}. We denote by p_i the i-th projection and by p_{ij} the (i,j)-th projection (*e.g.*, p_{13} is the projection from $X_1 \times X_1^{\mathrm{a}} \times X_2$ to $X_1 \times X_2$).
2. We write $\mathcal{A}_i$ and $\mathcal{A}_{ij^{\mathrm{a}}}$ instead of $\mathcal{A}_{X_i}$ and $\mathcal{A}_{X_i \times X_j^{\mathrm{a}}}$ and similarly with other products.

Let $f : X_1 \to X_2$ be a morphism of smooth complex algebraic varieties and assume X_2 is endowed with a DQ-algebroid stack $\mathcal{A}_2$. The usual push-forward/pullback adjunction $f^{-1} \dashv f_*$

$$\mathscr{D}(f^{-1}\mathcal{A}_2) \underset{f^{-1}}{\overset{f_*}{\rightleftarrows}} \mathscr{D}(\mathcal{A}_2)$$

induces an adjunction

$$\mathscr{D}_{\mathrm{cc}}(f^{-1}\mathcal{A}_2) \underset{\hat{f}^{-1}}{\overset{f_*}{\rightleftarrows}} \mathscr{D}(\mathcal{A}_2).$$

Indeed, the operation f_* preserves cohomological completeness by [KS12, Proposition 1.5.12]. Thus f_* induces a functor

$$f_* : \mathscr{D}_{\mathrm{cc}}(f^{-1}\mathcal{A}_2) \to \mathscr{D}_{\mathrm{cc}}(\mathcal{A}_2).$$

Its left adjoint is the cohomological completion

$$\hat{f}^{-1} : \mathscr{D}_{\mathrm{cc}}(f^{-1}\mathcal{A}_2) \to \mathscr{D}_{\mathrm{cc}}(\mathcal{A}_2)$$

of f^{-1}, which we will typically denote f^{-1} instead of $\hat{f}^{-1}$.

The relative tensor products of DQ-modules is the $\mathbb{C}^\hbar$-linear functor

$$-\underset{\mathcal{A}_2}{\otimes}- : \mathscr{D}(\mathcal{A}_{12^{\mathrm{a}}}) \times \mathscr{D}(\mathcal{A}_{23^{\mathrm{a}}}) \to \mathscr{D}(p_{13}^{-1}\mathcal{A}_{13^{\mathrm{a}}}),\ (\mathcal{K}_1, \mathcal{K}_2) \mapsto p_{12}^{-1}\mathcal{K}_1 \underset{p_{12}^{-1}\mathcal{A}_{12^{\mathrm{a}}}}{\otimes} \mathcal{A}_{123} \underset{p_{23^{\mathrm{a}}}^{-1}\mathcal{A}_{23}}{\otimes} p_{23}^{-1}\mathcal{K}_2$$

Its cohomological completion induces the functor

$$-\underset{\mathcal{A}_2}{\widehat{\otimes}}- : \mathscr{D}_{\mathrm{cc}}(\mathcal{A}_{12^{\mathrm{a}}}) \times \mathscr{D}_{\mathrm{cc}}(\mathcal{A}_{23^{\mathrm{a}}}) \to \mathscr{D}_{\mathrm{cc}}(p_{13}^{-1}\mathcal{A}_{13^{\mathrm{a}}}).$$

Combining the above operations, we define the composition of cohomologically complete kernels.

Definition 1.34. We write $-\underset{2}{\circ}- : \mathscr{D}_{\mathrm{cc}}(\mathcal{A}_{12^{\mathrm{a}}}) \times \mathscr{D}_{\mathrm{cc}}(\mathcal{A}_{23^{\mathrm{a}}}) \to \mathscr{D}_{\mathrm{cc}}(\mathcal{A}_{13^{\mathrm{a}}})$ for the pushforward of the relative tensor product functor, $(\mathcal{K}_1, \mathcal{K}_2) \mapsto \mathcal{K}_1 \underset{2}{\circ} \mathcal{K}_2 := p_{13*}(\mathcal{K}_1 \underset{\mathcal{A}_2}{\widehat{\otimes}} \mathcal{K}_2)$.

The functor $-\underset{2}{\circ}-$ is $\mathbb{C}^\hbar$-bilinear by construction and we have the following result.

Lemma 1.35. *Let $\mathcal{K}_i \in \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_{i(i+1)^{\mathrm{a}}})$ $(i = 1, 2)$. The kernel $\mathcal{K}_1 \underset{2}{\circ} \mathcal{K}_2$ is an object of $\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_{13^{\mathrm{a}}})$.*

If $\mathcal{K} \in \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_{12^{\mathrm{a}}})$, the above lemma implies that the following functor is well-defined:

$$\Phi_{\mathcal{K}} : \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_2) \to \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_1), \quad \mathscr{M} \mapsto \mathcal{K} \underset{2}{\circ} \mathscr{M} = p_{1*}(\mathcal{K} \underset{p_2^{-1}\mathcal{A}_2}{\widehat{\otimes}} \hat{p}_2^{-1}\mathscr{M}).$$

1.5.2 Integral representations

One of the aims of [GP20] was to establish an integral representation theorem for coherent DQ-modules, as already mentioned, one of the difficulties is that coherent DQ-modules are not necessarily compact in $\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X)$. Our strategy is to first prove an integral representation theorem for qcc-modules following the approach of [BZFN10]. Then, we show that coherent DQ-modules corresponds to integral transforms preserving compact object and combining these two results with the fact that a Fourier-Mukai transform for coherent DQ-modules on proper varieties has a right adjoint we get the desired result.

Integral representation for qcc modules

If X and Y are smooth complex algebraic varieties endowed with DQ-algebroid stacks $\mathcal{A}_X$ and $\mathcal{A}_Y$ then the variety $X \times Y$ is canonically equipped with a DQ-algebroid stack denoted $\mathcal{A}_{X\times Y}$. Following [KS12, p.68], we define the functor $\underline{\boxtimes}\colon \mathscr{D}(\mathcal{A}_X) \times \mathscr{D}(\mathcal{A}_Y) \to \mathscr{D}(\mathcal{A}_{X\times Y})$ by

$$(\mathcal{M}, \mathcal{N}) \mapsto \mathcal{A}_{X\times Y} \underset{\mathcal{A}_X \boxtimes \mathcal{A}_Y}{\otimes} (\mathcal{M} \underset{\mathbb{C}^\hbar}{\boxtimes} \mathcal{N}).$$

This induces a $\mathscr{D}(\mathbb{C}^\hbar)$-bilinear functor

$$\underline{\boxtimes}\colon \mathscr{D}_{\mathrm{gqcoh}}(\mathcal{A}_X) \times \mathscr{D}_{\mathrm{gqcoh}}(\mathcal{A}_Y) \to \mathscr{D}_{\mathrm{gqcoh}}(\mathcal{A}_{X\times Y}).$$

Applying the cohomological completion functor to the above functor we obtain

$$\widehat{\underline{\boxtimes}}\colon \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X) \times \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_Y) \longrightarrow \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_{X\times Y})$$

which by $\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)$-linearity provides the functor

$$\widehat{\underline{\boxtimes}}\colon \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X) \underset{\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)}{\otimes} \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_Y) \longrightarrow \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_{X\times Y}). \tag{1.5.1}$$

The next result is a non-commutative version of [BZFN10, Theorem 4.7].

Theorem 1.36 ([GP20, Theorem 4.15])**.** *The functor* (1.5.1) *is an equivalence.*

We also obtained.

Proposition 1.37. *The* $\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)$*-module* $\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ *is dualizable and its dual is* $\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_{X^{\mathrm{a}}})$*.*

The above proposition is the DQ-module counterpart of [BZFN10, Corollary 4.8] The unicity of duals implies that the evaluation map

$$\varepsilon\colon \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_{X^{\mathrm{a}}}) \underset{\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)}{\otimes} \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X) \to \mathscr{D}_{\mathrm{cc}}(\mathbb{C}^\hbar)$$

induces an equivalence

$$\gamma\colon \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_{X^{\mathrm{a}}}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{L}}_{\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^{\hbar})}(\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X), \mathscr{D}_{\mathrm{cc}}(\mathbb{C}^{\hbar})).$$

With these results at hand, we define the functor

$$\Phi_{(-)}\colon \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_{Y\times X^{\mathrm{a}}}) \longrightarrow \mathrm{Fun}^{\mathrm{L}}_{\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^{\hbar})}(\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X), \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_Y)). \tag{1.5.2}$$

as the composition of the equivalences (1.5.3), (1.5.4) and (1.5.5).

$$\widehat{\underline{\boxtimes}}^{-1}\colon \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_{Y\times X^{\mathrm{a}}}) \longrightarrow \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_Y) \underset{\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^{\hbar})}{\otimes} \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_{X^{\mathrm{a}}}) \tag{1.5.3}$$

$$\mathrm{id} \otimes \gamma\colon \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_Y) \underset{\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^{\hbar})}{\otimes} \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_{X^{\mathrm{a}}}) \xrightarrow{\sim} \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_Y) \underset{\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^{\hbar})}{\otimes} \mathrm{Fun}^{\mathrm{L}}_{\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^{\hbar})}(\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X), \mathscr{D}_{\mathrm{cc}}(\mathbb{C}^{\hbar})) \tag{1.5.4}$$

$$\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_Y) \underset{\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^{\hbar})}{\otimes} \mathrm{Fun}^{\mathrm{L}}_{\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^{\hbar})}(\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X), \mathscr{D}_{\mathrm{cc}}(\mathbb{C}^{\hbar})) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{L}}_{\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^{\hbar})}(\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X), \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_Y)) \tag{1.5.5}$$

Hence we obtain the DQ analogue of [Toë07, Theorem 8.9].

Theorem 1.38 ([GP20, Theorem 4.19])**.** *Let X and Y be two smooth algebraic varieties endowed with DQ-algebroid stacks $\mathcal{A}_X$ and $\mathcal{A}_Y$. There is an equivalence of ∞-categories*

$$\Phi_{(-)}\colon \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_{Y\times X^{\mathrm{a}}}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{L}}_{\mathscr{D}_{\mathrm{cc}}(\mathbb{C}^{\hbar})}(\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X), \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_Y)). \tag{1.5.6}$$

It is possible to compute explicitly $\Phi_{(-)}$ on kernels of the form $\mathcal{M} \widehat{\underline{\boxtimes}} \mathcal{N}$. In this case, we get that

$$\Phi_{(\mathcal{M}\widehat{\underline{\boxtimes}}\mathcal{N})} \simeq p_{Y*}((\mathcal{M}\widehat{\underline{\boxtimes}}\mathcal{N}) \underset{p_X^{-1}\mathcal{A}_X}{\widehat{\otimes}} p_X^{-1}(-)).$$

Moreover, the category $\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_{X^{\mathrm{a}}\times Y})$ is generated under colimits by objects of the form $\mathcal{M} \widehat{\underline{\boxtimes}} \mathcal{N}$ and $\Phi_{(-)}$ is a functor in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. Hence, it commutes with colimits. This implies that for every $\mathcal{K} \in \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_{X^a\times Y})$,

$$\Phi_{\mathcal{K}} \simeq p_{Y*}(\mathcal{K} \underset{p_X^{-1}\mathcal{A}_X}{\widehat{\otimes}} p_X^{-1}(-)). \tag{1.5.7}$$

A first consequence of these results is the following.

Corollary 1.39 ([GP20, Corollary 4.21])**.** *Let X and Y be two smooth algebraic varieties endowed with DQ-algebroid stacks $\mathcal{A}_X$ and $\mathcal{A}_Y$. If $\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X) \simeq \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_Y)$ in $\mathrm{Mod}_{\mathscr{D}(\mathbb{C}^\hbar)}$ then $\mathscr{D}_{\mathrm{qcoh}}(\mathcal{O}_X) \simeq \mathscr{D}_{\mathrm{qcoh}}(\mathcal{O}_Y)$ in $\mathrm{Mod}_{\mathscr{D}(\mathbb{C})}$.*

We now focus our attention on the functors that preserve the compact objects of the qcc-modules. Recall that

Theorem 1.40 ([Pet14, Theorem 3.12])**.** *Let X (resp. Y) be a smooth complex proper algebraic variety endowed with a DQ-algebroid $\mathcal{A}_X$ (resp. $\mathcal{A}_Y$). Let $\mathcal{K} \in \mathrm{D}_{\mathrm{qcc}}(\mathcal{A}_{Y\times X^{\mathrm{a}}})$. The functor $\Phi_{\mathcal{K}} : \mathrm{D}_{\mathrm{qcc}}(\mathcal{A}_X) \to \mathrm{D}_{\mathrm{qcc}}(\mathcal{A}_Y)$ preserves compact objects if and only if $\mathcal{K}$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_{Y\times X^{\mathrm{a}}})$.*

Using Theorems 1.38 and 1.40, we obtain the following result

Theorem 1.41 ([GP20, Theorem 4.22])**.** *Let X and Y be two smooth and proper algebraic varieties endowed with DQ-algebroid stacks $\mathcal{A}_X$ and $\mathcal{A}_Y$. The functor*

$$\Phi_{(-)} \colon \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_{Y\times X^{\mathrm{a}}}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{L}}_{\mathscr{D}(\mathbb{C}^\hbar)}(\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X), \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_Y))$$

in (1.5.2) *induces by restriction an equivalence*

$$\widetilde{\Phi}_{(-)} \colon \mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_{Y\times X^{\mathrm{a}}}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{L}}_{\mathscr{D}(\mathbb{C}^\hbar),\omega}(\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X), \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_Y)).$$

Now, using the equivalence of ∞-categories (see [BZFN10, BGT13])

Lemma 1.42. *The functor $(-)^\omega : \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega} \to \mathrm{Cat}^{\mathrm{ex}}_{\infty,\mathrm{idem}}$ is an equivalence of symmetric monoidal ∞-categories. In particular, if R is a commutative ring (or commutative ring spectrum), then $(-)^\omega : \mathrm{Mod}_{\mathscr{D}(R)}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}) \to \mathrm{Mod}_{\mathscr{D}^\omega(R)}(\mathrm{Cat}^{\mathrm{ex}}_{\infty,\mathrm{idem}})$ is an equivalence of ∞-categories.*

we obtain the following result.

Corollary 1.43 ([GP20, Corollary 4.23])**.** *Let X and Y be two smooth and proper algebraic varieties endowed with DQ-algebroid stacks $\mathcal{A}_X$ and $\mathcal{A}_Y$. Then the functor $\Phi_{(-)}$ induces an equivalence*

$$\mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_{Y\times X^{\mathrm{a}}}) \simeq \mathrm{Fun}_{\mathscr{D}^\omega(\mathbb{C}^\hbar)}(\mathscr{D}^\omega_{\mathrm{qcc}}(\mathcal{A}_X), \mathscr{D}^\omega_{\mathrm{qcc}}(\mathcal{A}_Y))$$

Theorem 1.44. *Let X and Y be two smooth and proper algebraic varieties endowed with DQ-algebroid stacks $\mathcal{A}_X$ and $\mathcal{A}_Y$. Then the functor $\Phi_{(-)}$ induces an equivalence*

$$\mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_{Y\times X^{\mathrm{a}}}) \simeq \mathrm{Fun}^{\mathrm{L}}_{\mathscr{D}^\omega(\mathbb{C}^\hbar)}(\mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X), \mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_Y)) \subset \mathrm{Fun}_{\mathscr{D}^\omega(\mathbb{C}^\hbar)}(\mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X), \mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_Y))$$

where $\mathrm{Fun}^{\mathrm{L}}_{\mathscr{D}^\omega(\mathbb{C}^\hbar)}(\mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X), \mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_Y))$ designate the full $\mathscr{D}^\omega(\mathbb{C}^\hbar)$-linear subcategory of the category $\mathrm{Fun}_{\mathscr{D}^\omega(\mathbb{C}^\hbar)}(\mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X), \mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_Y))$, spanned by the left adjoint functors.

Proof. This proof is extracted from [GP20] and we reproduced it here as it relies several of the previous results.

Recall that $\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X) \simeq \mathrm{Ind}(\mathscr{D}^{\omega}_{\mathrm{qcc}}(\mathcal{A}_X))$. We denote by $j\colon \mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X) \subset \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X)$ the fully faithful inclusion of the ∞-category of coherent DQ-modules into the ∞-category of qcc DQ-modules. Passing to Ind-objects induces an equivalence of ∞-categories

$$\mathrm{Fun}_{\mathscr{D}^{\omega}(\mathbb{C}^{\hbar})}(\mathscr{D}^{\omega}_{\mathrm{qcc}}(\mathcal{A}_X), \mathscr{D}^{\omega}_{\mathrm{qcc}}(\mathcal{A}_Y)) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{L}}_{\mathscr{D}(\mathbb{C}^{\hbar}),\omega}(\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X), \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_Y)),$$

and j induces a functor

$$\mathrm{Fun}^{\mathrm{L}}_{\mathscr{D}(\mathbb{C}^{\hbar}),\omega}(\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X), \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_Y)) \xrightarrow{j^*} \mathrm{Fun}_{\mathscr{D}^{\omega}(\mathbb{C}^{\hbar})}(\mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X), \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_Y)).$$

We know by Theorem 1.41, that the objects of $\mathrm{Fun}^{\mathrm{L}}_{\mathscr{D}(\mathbb{C}^{\hbar}),\omega}(\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X), \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_Y))$ correspond to Fourier-Mukai transforms with coherent kernels. Hence, the restriction of such a functor to $\mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X)$ induces a $\mathscr{D}^{\omega}(\mathbb{C}^{\hbar})$-linear functor from $\mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X)$ to $\mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_Y)$. Furthermore, such a Fourier-Mukai transform $\Phi_{\mathcal{K}}\colon \mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X) \to \mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_Y)$ has a right adjoint, as shown in the proof of [Pet14, Proposition 3.15]. Thus j_* factors through the full subcategory

$$\mathrm{Fun}^{\mathrm{L}}_{\mathscr{D}^{\omega}(\mathbb{C}^{\hbar})}(\mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X), \mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_Y)) \subset \mathrm{Fun}_{\mathscr{D}^{\omega}(\mathbb{C}^{\hbar})}(\mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X), \mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_Y))$$

and we get a functor

$$\begin{aligned}\alpha\colon \mathrm{Fun}_{\mathscr{D}^{\omega}(\mathbb{C}^{\hbar})}(\mathscr{D}^{\omega}_{\mathrm{qcc}}(\mathcal{A}_X), \mathscr{D}^{\omega}_{\mathrm{qcc}}(\mathcal{A}_Y)) &\xrightarrow{\mathrm{Ind}} \mathrm{Fun}^{\mathrm{L}}_{\mathscr{D}(\mathbb{C}^{\hbar}),\omega}(\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X), \mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_Y))\\ &\xrightarrow{j^*} \mathrm{Fun}^{\mathrm{L}}_{\mathscr{D}^{\omega}(\mathbb{C}^{\hbar})}(\mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X), \mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_Y)).\end{aligned}$$

Hence, $\alpha := j^* \circ \mathrm{Ind}$. The fully faithful inclusion $i\colon \mathscr{D}^{\omega}_{\mathrm{qcc}}(\mathcal{A}_X) \subset \mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X)$ induces a functor

$$\mathrm{Fun}^{\mathrm{L}}_{\mathscr{D}^{\omega}(\mathbb{C}^{\hbar})}(\mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X), \mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_Y)) \xrightarrow{i^*} \mathrm{Fun}_{\mathscr{D}^{\omega}(\mathbb{C}^{\hbar})}(\mathscr{D}^{\omega}_{\mathrm{qcc}}(\mathcal{A}_X), \mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_Y)) \tag{1.5.8}$$

If $F \in \mathrm{Fun}^{\mathrm{L}}_{\mathscr{D}^{\omega}(\mathbb{C}^{\hbar})}(\mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X), \mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_Y))$ and $\mathscr{M} \in \mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X)$ is of $\hbar$-torsion, then the $\mathscr{D}^{\omega}(\mathbb{C}^{\hbar})$-linearity of F implies that $F(\mathscr{M})$ is again of $\hbar$-torsion. It follows from Theorem 1.29 that the functor (1.5.8) factors through $\mathrm{Fun}_{\mathscr{D}^{\omega}(\mathbb{C}^{\hbar})}(\mathscr{D}^{\omega}_{\mathrm{qcc}}(\mathcal{A}_X), \mathscr{D}^{\omega}_{\mathrm{qcc}}(\mathcal{A}_Y))$ and we get a functor

$$\beta\colon \mathrm{Fun}^{\mathrm{L}}_{\mathscr{D}^{\omega}(\mathbb{C}^{\hbar})}(\mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X), \mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_Y)) \xrightarrow{i^*} \mathrm{Fun}_{\mathscr{D}^{\omega}(\mathbb{C}^{\hbar})}(\mathscr{D}^{\omega}_{\mathrm{qcc}}(\mathcal{A}_X), \mathscr{D}^{\omega}_{\mathrm{qcc}}(\mathcal{A}_Y)). \tag{1.5.9}$$

It is straightforward to check that $\beta \circ \alpha(F) = \mathrm{Ind}(F) \circ j \circ i \simeq F$ and $\alpha \circ \beta(G) \circ i \simeq \mathrm{Ind}(G \circ i) \circ j \circ i \simeq G \circ i$. Moreover, any object $\mathscr{M}$ of $\mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X)$ can be written as a filtered colimit of objects of $\mathscr{D}^{\omega}_{\mathrm{qcc}}(\mathcal{A}_X)$ and these colimits are also colimits in $\mathscr{D}_{\mathrm{qcc}}(\mathcal{A}_X)$.

The functor G commutes with colimits by hypothesis and $\alpha\circ\beta(G)$ commutes with colimits of the above type. Hence $\alpha \circ \beta(G) \simeq G$. $\square$

Corollary 1.45. *Let X and Y be two smooth proper algebraic varieties endowed with DQ-algebroid stacks $\mathcal{A}_X$ and $\mathcal{A}_Y$. If $\mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_X) \simeq \mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_Y)$ in $\mathrm{Mod}_{\mathscr{D}^\omega(\mathbb{C}^\hbar)}(\mathrm{Cat}^{\mathrm{ex}}_{\infty,\mathrm{idem}})$ then $\mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_X) \simeq \mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_Y)$ in $\mathrm{Mod}_{\mathscr{D}^\omega(\mathbb{C})}(\mathrm{Cat}^{\mathrm{ex}}_{\infty,\mathrm{idem}})$.*

Proof. An equivalence in $\mathrm{Mod}_{\mathscr{D}^\omega(\mathbb{C}^\hbar)}(\mathrm{Cat}^{\mathrm{ex}}_{\infty,\mathrm{idem}})$ is a left adjoint functor. Hence, by Theorem 1.44, it is a Fourier-Mukai functor $\Phi_{\mathscr{K}}$. It follows by [Pet14, Theorem 3.16] that $\Phi_{\mathrm{gr}_\hbar \mathcal{K}} \colon \mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_X) \to \mathscr{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_Y)$ is an equivalence. $\square$

Chapter 2

Quantization of spectral curves

2.1 Introduction

Spectral curves initially arose in the study of certain integrable systems as the zero locus of families of characteristic polynomials. Their quantization is currently investigated due to their links with topological recursion and problems of enumerative geometry where certain generating functions appear as solutions of some specific quantum curves (see [DBMN$^+$17] for the case of the exponential generating function of Gromov-Witten invariants on $\mathbb{P}^1$ or [LMS18] for the case of the generating functions of simple Hurwitz numbers on an arbitrary base curve).

Quantum curves are often defined in terms of modules over the Rees algebra of the sheaf of holomorphic differential operators filtered by the order [DHS09, DM14]. Though, this definition is well suited to study the quantization of spectral curves of Higgs bundles, it does not provide a setting flexible enough to capture several examples of quantum curves as for instance those appearing in the quantization of the A-polynomial [BE15, Dim13, GS12b].

Around 2015, an actively studied question was to understand the relation between the quantum curve quantizing a given spectral curve and the topological recursion taking this spectral curve as a parameter (among others). The topological recursion produces under certain condition the WKB expansion of the aforementioned quantum curve [BE17]. Moreover, at that time, the relation with deformation quantization was unclear and is now much better understood thanks to [KS18b]. The conjectural relation between quantum curves and Topological Recursion [EO07] suggested that it should always be possible to quantize the spectral curve associated to a Higgs bundle. The existence of such a quantization was unclear in 2015 when [Pet18c] was written. This situation was usually studied by using modules over the Rees algebra of differential operators filtered by the order. We approached this question from the standpoint of deformation quantization as spectral curves can be considered as Lagrangian submanifolds of symplectic surfaces and the theory of DQ-modules is well suited to study quantization of Lagrangians. Nonetheless, here we wished to quantize not only the spectral curve but also the section defining it. Hence, the classical results concerning the deformation quantization of Lagrangian submanifolds [DS07, BGKP15, BC17] do not seem to apply. Quantum curves are often sections of a sheaf of operator $\mathcal{R}$ on a Riemann surface Σ and we remarked in [Pet18c] that it is usually possible find a Lagrangian fibration $p\colon X \to \Sigma$, together with a DQ algebra $\mathcal{A}_X$ on X such that the sheaf $p^{-1}\mathcal{R}$ is a flat subalgebra of $\mathcal{A}_X$. This situation is reminiscent of the relation between $\mathcal{D}$-modules and microdifferential modules and suggests to define a notion of microlocal quantum curves. We showed that given a spectral curve associated with a Higgs bundle on a Riemann surface of genus $g \geq 2$, it can always be quantized by a microlocal quantum curve (which immediately implies that the spectral curve is quantized by a holonomic DQ-module). The usual strategy to quantize spectral curve was to provide a local model of the quantization and then try to globalize it. Here, instead of trying to globalize a local construction, we reduce the problem to a question concerning the vanishing of a certain cohomology group. This leads to a particularly simple proof. Our result suggests that one of the problem is not the ability to quantize a spectral curve but the ability to identify quantum curves of interest among the possible quantization of a given spectral curve. It is also natural to wonder in which extent the analogy between the quantum curve framework and the microdifferential/$\mathcal{D}$-modules setting can be developed.

2.2 Quantization of the cotangent bundle

There is a canonical quantization by a DQ-algebra of the cotangent bundle of a complex manifold. It was constructed in [PS04]. Here, we review their construction as we will use it in this chapter and the next one.

Let M be a complex manifold. We write $X := T^*M$ for the cotangent bundle of M, and $\pi\colon T^*M \to M$ for the canonical projection. The manifold X is equipped with the sheaf $\widehat{\mathcal{E}}_X$ of formal microdifferential operators. This is a filtered, conic sheaf of $\mathbb{C}$-algebras. We denote by $\widehat{\mathcal{E}}_X(0)$ the subsheaf of $\widehat{\mathcal{E}}_X$ formed by the operators of order $m \leq 0$. These sheaves were introduced in [SKK73]. The reader can consult [Sch85] for an introduction to the theory of microdifferential modules.

Let $\mathbb{C}$ be the complex line endowed with the coordinate t and denote by $(t;\tau)$ the associated symplectic coordinate on $T^*\mathbb{C}$. We set

$$\widehat{\mathcal{E}}_{T^*(M\times\mathbb{C}),\hat{t}}(0) = \{P \in \widehat{\mathcal{E}}_{T^*M}; [P, \partial_t] = 0\}.$$

We consider the following open subset of $T^*(M\times\mathbb{C})$

$$T^*_{\tau\neq 0}(M\times\mathbb{C}) = \{(x,t;\xi,\tau) \in T^*(M\times\mathbb{C}); \tau \neq 0\}$$

and the morphism

$$\rho\colon T^*_{\tau\neq 0}(M\times\mathbb{C})\to T^*M,\ (x,t;\xi,\tau)\mapsto(x;\xi/\tau).$$

We obtain the $\mathbb{C}^\hbar_X$-algebra

$$\widehat{\mathcal{W}}_X(0):=\rho_*(\widehat{\mathcal{E}}_{T^*(M\times\mathbb{C}),\hat{t}}(0)|_{T^*_{\tau\neq0}(M\times\mathbb{C})}) \tag{2.2.1}$$

where $\hbar$ acts as τ^{-1}. A section P of $\widehat{\mathcal{W}}_X(0)$ can be written in a local symplectic coordinate system $(x_1,\dots,x_n,u_1,\dots,u_n)$ as

$$P=\sum_{j\leq 0} f_{-j}(x,u_i)\tau^j,\ f_{-j}\in\mathcal{O}_X,\ j\in\mathbb{Z}.$$

Setting $\hbar=\tau^{-1}$, we obtain

$$P=\sum_{k\geq 0} f_k(x,u_i)\hbar^k,\ f_k\in\mathcal{O}_X,\ k\in\mathbb{N}.$$

We denote by $\widehat{\mathcal{W}}_X$ the localization of $\widehat{\mathcal{W}}_X(0)$ with respect to the parameter $\hbar$.

There is the following commutative diagram of morphisms of algebras.

$$\begin{array}{ccccc} \pi^{-1}\mathcal{D}_M & \hookrightarrow & \widehat{\mathcal{E}}_X & \overset{\iota}{\hookrightarrow} & \widehat{\mathcal{W}}_X \\ \uparrow & & \uparrow & & \uparrow \\ \pi^{-1}\mathcal{O}_M & \hookrightarrow & \widehat{\mathcal{E}}_X(0) & \hookrightarrow & \widehat{\mathcal{W}}_X(0) \end{array} \tag{2.2.2}$$

In a symplectic affine chart $(x_1,\dots,x_n;\xi_1,\dots,\xi_n)$, the algebra map $\iota:\widehat{\mathcal{E}}_X\to\widehat{\mathcal{W}}_X$ amount to $x_i\mapsto x_i$, $\partial_{x_i}\mapsto\hbar^{-1}u_i$.

2.3 Microlocal quantum curves

2.3.1 A general setting

In this section, we introduce the notion of microlocal quantum curve. We first set a few notations. If E is a vector bundle on a complex manifold X and $s\colon X\to E$ is a section of E, we denote be X_s the analytic zero subscheme of s and by $Z(s)$ the set theoretic zero locus of s.

Definition 2.1. Let X be a complex symplectic manifold and let $\mathcal{A}_X$ be a DQ-algebra the associated Poisson structure of which is the symplectic structure of X. A polarization $\mathcal{P}$ of $(X,\mathcal{A}_X)$ is the data of

(i) a holomorphic fiber bundle $\pi:X\to\Sigma$ such that for every $y\in\Sigma$, $\pi^{-1}(y)$ is a

Lagrangian submanifold of X,

(ii) a Lagrangian immersion $\iota : \Sigma \to X$ such that $\pi \circ \iota = \mathrm{id}_\Sigma$,

(iii) a morphism of algebra $\phi\colon \pi^{-1}\mathcal{O}_\Sigma \to \mathcal{A}_X$,

(iv) An $\mathcal{A}_X$-module $\mathcal{Q}$ simple along Σ.

Furthermore, given a sheaf of algebra $\mathcal{R}$ on Σ such that $\mathcal{O}_\Sigma$ is a subalgebra of $\mathcal{R}$, we say that a polarization $\mathcal{P}$ is $\mathcal{R}$-microlocal (or microlocal if there is no risk of confusion), if there is a flat morphism of algebras $\psi\colon \pi^{-1}\mathcal{R} \to \mathcal{A}_X$ such that the diagram

$$\begin{array}{ccc} \pi^{-1}\mathcal{O}_\Sigma & \xrightarrow{\quad\phi\quad} & \mathcal{A}_X \\ & \searrow \quad\quad \nearrow_{\psi} & \\ & \pi^{-1}\mathcal{R} & \end{array}$$

is commutative.

Remark 2.2. 1. The notion of $\mathcal{R}$-microlocal polarization highlights the analogy between on one hand DQ-algebras and various algebras of quantum curves and on the other hand the algebra of microdifferential operators together with the algebra of differential operators on Σ.

2. In practice, it is often the case that $\mathcal{Q} \simeq \mathcal{O}_\Sigma^\hbar$ as $\mathbb{C}_X^\hbar$-modules in Definition 2.1 (iv).

Definition 2.3. Let X be a complex symplectic manifold and let $\mathcal{A}_X$ be a DQ-algebra the associated Poisson structure of which is the symplectic structure of X, $\mathcal{P}$ a polarization of $(X, \mathcal{A}_X)$ and let $\mathcal{L}$ be an $\mathcal{O}_\Sigma$-line bundle. The sheaf of $\mathcal{L}$-microlocal quantum curves is the sheaf

$$\mathcal{A}_X^\mathcal{L} := \mathcal{A}_X \otimes_{\pi^{-1}\mathcal{O}_\Sigma} \pi^{-1}\mathcal{L}.$$

Notice that $\mathcal{A}_X^\mathcal{L}$ is a coherent, $\hbar$-complete $\mathcal{A}_X$-module without $\hbar$-torsion such that $\mathcal{A}_X^\mathcal{L}/\hbar\mathcal{A}_X^\mathcal{L} \simeq \pi^*\mathcal{L}$. We denote by $\sigma_0\colon \mathcal{A}_X^\mathcal{L} \to \pi^*\mathcal{L}$ the canonical projection. The module $\mathcal{A}_X^\mathcal{L}$ fits in the following exact sequence

$$0 \to \hbar\mathcal{A}_X^\mathcal{L}(0) \to \mathcal{A}_X^\mathcal{L}(0) \overset{\sigma_0}{\to} \pi^*\mathcal{L} \to 0.$$

If s is a section of $\pi^*\mathcal{L}$, a microlocal quantum curve quantizing s is a section $s_\hbar \in \mathcal{A}_X^\mathcal{L}$ such that $\sigma_0(s_\hbar) = s$. Such a section allows to define the following coherent $\mathcal{A}_X$-module

$$\mathcal{M} := \mathcal{A}_X^\mathcal{L}/(s_\hbar)$$

where $(s_\hbar)$ denotes the left submodules of $\mathcal{A}_X^\mathcal{L}$ generated by $s_\hbar$. Since $\mathcal{M}$ has no $\hbar$-torsion, we have

$$\mathrm{Supp}(\mathcal{M}) = \mathrm{Supp}(\mathcal{M}/\hbar\mathcal{M}) = \mathrm{Supp}(\pi^*\mathcal{L}/(s)) = Z(s).$$

The above setting provides the following simple result.

Proposition 2.4 ([Pet18c, Proposition3.1])**.** *Let X be a complex symplectic surface and let $\mathcal{A}_X$ be a DQ-algebra the associated Poisson structure of which is the symplectic structure of X, let $\mathcal{P}$ be a polarization of $(X, \mathcal{A}_X)$, $\mathcal{L}$ be a line bundle on Σ and $s \neq 0$ be a section of $\pi^*\mathcal{L}$. Assume that $\mathrm{H}^1(X, \mathcal{A}_X^{\mathcal{L}}) = 0$. Then, there exists a $\mathcal{L}$-microlocal quantum curve $s_\hbar$ quantizing s and a coherent $\mathcal{A}_X$-module $\mathcal{M}$ without $\hbar$-torsion supported by $Z(s)$ and such that $\mathcal{M}/\hbar\mathcal{M} \simeq \pi^*\mathcal{L}/(s)$.*

2.3.2 Quantization of spectral curves of Higgs bundles

Let Σ be a Riemann surface, $X = T^*\Sigma$, $\pi\colon T^*\Sigma \to \Sigma$ be the canonical projection and η the Liouville form of $T^*\Sigma$. We endow X with the DQ-algebra $\widehat{\mathcal{W}}_X(0)$. The data of π, of the inclusion of the zero section $\Sigma \hookrightarrow T^*\Sigma$, of the morphism $\pi^{-1}\mathcal{O}_\Sigma \to \widehat{\mathcal{W}}_X(0)$ from Diagram (2.2.2) and the $\widehat{\mathcal{W}}_X(0)$-module $\mathcal{O}_\Sigma^\hbar$ specify a polarization of $(X, \widehat{\mathcal{W}}_X(0))$.

Let $\mathcal{L}$ be a line bundle on Σ and consider the $\widehat{\mathcal{W}}_X(0)$-module $\widehat{\mathcal{W}}_X^{\mathcal{L}}(0)$. We obtained the following result

Lemma 2.5 ([Pet18c, Lemma 3.10])**.** *Let Σ be a compact Riemann surface and $\mathcal{L}$ be a line bundle such that $\mathrm{H}^1(\Sigma, \mathcal{L}) = 0$. Then,*

$$\mathrm{H}^i(X, \widehat{\mathcal{W}}_X^{\mathcal{L}}(0)) \simeq 0, \text{ for } i > 0.$$

We recall a few definitions associated with the notion of Higgs bundle.

Definition 2.6. (i) A Higgs bundle on Σ is a pair $(\mathcal{E}, \phi)$ where $\mathcal{E}$ is a locally free $\mathcal{O}_\Sigma$-module of finite rank and $\phi \in \mathrm{H}^0(\Sigma, \mathcal{E}\mathrm{nd}(\mathcal{E}) \otimes \Omega_\Sigma^1)$. The section ϕ is called the Higgs field of $(\mathcal{E}, \phi)$.

(ii) Let $(\mathcal{E}, \phi)$ be a Higgs bundle on Σ. The characteristic polynomial of ϕ is the section $\det(\pi^*\phi - \eta \otimes \mathrm{id}_{\pi^*\mathcal{E}}) \in \Gamma(X; \pi^*\Omega_\Sigma^{1\otimes r})$. We often write $\det(\pi^*\phi - \eta)$ instead of $\det(\pi^*\phi - \eta \otimes \mathrm{id}_{\pi^*\mathcal{E}})$ for brevity.

(iii) The spectral curve of a Higgs bundle $(\mathcal{E}, \phi)$ is the zero locus of the section $\det(\pi^*\phi - \eta)$.

We now apply the above results to the quantization of spectral curve

Theorem 2.7. *Let Σ be a compact Riemann surface of genus $g \geq 2$ and $(\mathcal{E}, \phi)$ be a Higgs bundle of rank $r \geq 2$ on Σ. Then there exists a $\Omega_\Sigma^{1\otimes r}$-microlocal quantum curve $s_\hbar$ quantizing $\det(\pi^*\phi - \eta)$. The section $s_\hbar$ induces the holonomic $\widehat{\mathcal{W}}_X(0)$-module*

$$\mathcal{M} = \widehat{\mathcal{W}}_X^{\Omega_\Sigma^{1\otimes r}}(0)/\widehat{\mathcal{W}}_X(0)s_\hbar.$$

The module $\mathcal{M}$ *satisfies* $\mathcal{M}/\hbar\mathcal{M} \simeq \pi^*\Omega_\Sigma^{\otimes r}/(\det(\pi^*\phi - \eta))$ *(in particular* $\mathrm{Supp}(\mathcal{M}) = Z(\det(\pi^*\phi - \eta))$*). Moreover, if the analytic space* $X_{\det(\pi^*\phi-\eta)}$ *is smooth, then* $\mathcal{M}$ *is a simple* $\widehat{\mathcal{W}}_X(0)$*-module along* $X_{\det(\pi^*\phi-\eta)}$*.*

Proof. By Proposition 2.4, it is sufficient to verify that $\mathrm{H}^1(X, \widehat{\mathcal{W}}_X^{\Omega_\Sigma^{1\otimes r}}(0)) \simeq 0$. We know by Lemma 2.5 that this will be the case as soon as $\mathrm{H}^1(\Sigma, \Omega_\Sigma^{1\otimes r}) \simeq 0$. A sufficient condition for this is that

$$\deg \Omega_\Sigma^{1\otimes r} > \deg \Omega_\Sigma^1$$

where deg designate the degree of a line bundle. The above inequality is equivalent to

$$r(2g-2) > 2g-2$$

which is satisfied as soon as $r \geq 2$ and $g \geq 2$. □

Remark 2.8. The problem of quantizing a Lagrangian submanifold via DQ-modules has been thoroughly studied in [DS07, BGKP15, BC17]. Nonetheless, these result are not readily applicable to our situation as we are not only quantizing the Lagrangian but the section defining it.

2.3.3 Some examples of microlocal polarizations

Here, we present a few examples of microlocal polarization. We stress that this situation resemble the one observed with $\mathcal{D}$-modules and $\mathcal{E}$-modules. In these examples, the only technical point is to prove that the DQ-algebra is flat over the algebra of quantum curve $\mathcal{R}$. For that purpose, we rely over the following flatness criterion.

Theorem 2.9 ([Pet18c, Theorem 4.7])**.** *Let* X *be a complex manifold endowed with a DQ-algebra* $\mathcal{A}_X$ *and let* $\mathcal{B}_X$ *be a* $\mathcal{B}_X$*-coherent* $\mathbb{C}[\hbar]_X$ *sub-algebra of* $\mathcal{A}_X$*. Assume that* $\mathcal{A}_X/\hbar\mathcal{A}_X$ *is flat over* $\mathcal{B}_X/\hbar\mathcal{B}_X$*. Then,* $\mathcal{A}_X$ *is flat over* $\mathcal{B}_X$*.*

In the following examples of microlocal polarization, we do not specify explicitely the morphism ϕ of Definition 2.1 (iii) as we implicitly specify it when we precise the morphism ϕ appearing in this definition.

Example 2.10. Let $X = T^*\mathbb{C}$ endowed with its canonical symplectic structure, (x) is a coordinate on $\mathbb{C}$ and $(x; u)$ the associated symplectic coordinate system. We endow X with the star-algebra $(\mathcal{A}_X = \mathcal{O}_X^\hbar, \star)$ where

$$f \star g = \sum_{k \geq 0} \frac{\hbar^k}{k!} (\partial_u^k f)(\partial_x^k g).$$

We define the following microlocal polarization:

(i) $\Sigma = \mathbb{C}$,

(ii) $\pi\colon T^*\Sigma \to \Sigma$, $(x; u) \mapsto x$,

(iii) $\iota\colon \Sigma \hookrightarrow T^*\Sigma$ is the identification of Σ with the zero section of X,

(iv) $\mathcal{Q}$ is the quotient of $\mathcal{A}_X$ by the left ideal of $\mathcal{A}_X$ generated by $\hbar\partial_x$.

(v) Σ is equipped with the Rees-algebra of differential operators filtered by the order. Consider the sheaf $\mathcal{D}_\Sigma$ of holomorphic differential operators on Σ. For every $j \in \mathbb{N}$, we write $\mathcal{D}_\Sigma(j)$ for the j^{th} piece of the filtration by the order of $\mathcal{D}_\Sigma$ and $\mathcal{D}_\Sigma[\hbar]$ for $\mathcal{D}_\Sigma \underset{\mathbb{C}}{\otimes} \mathbb{C}[\hbar]$. The Rees algebra of $\mathcal{D}_\Sigma$ is the subsheaf of $\mathcal{D}_\Sigma[\hbar]$

$$R(\mathcal{D}_\Sigma) = \bigoplus_{j=0}^{\infty} \hbar^j \mathcal{D}_\Sigma(j).$$

There is a morphism of $\pi^{-1}\mathcal{O}_\Sigma$-algebras $\psi\colon \pi^{-1}R(\mathcal{D}_\Sigma) \to \mathcal{A}_X$ given by $x \mapsto x$, $\hbar\dfrac{\partial}{\partial x} \mapsto u$.

Proposition 2.11 ([Pet18c, Proposition 4.12])**.** *The algebra $\mathcal{A}_X$ is flat over $\pi^{-1}R(\mathcal{D}_\Sigma)$.*

In this setting, a standard example of spectral curve is the Airy curve defined by

$$u^2 - x = 0.$$

It is often quantized with the following quantum curve

$$(\hbar\frac{d}{dx})^2 - x.$$

Example 2.12. Consider the complex surface $X = \mathbb{C}^* \times \mathbb{C}^*$ with coordinate system (x_1, x_2) and symplectic form $\frac{dx_1 \wedge dx_2}{x_1 x_2}$. We endow X with the star-algebra $(\mathcal{A}_X = \mathcal{O}_X^\hbar, \star)$ where

$$f \star g = \sum_{k \geq 0} \frac{\hbar^k}{k!} (x_2\partial_{x_2})^k(f)(x_1\partial_{x_1})^k(g)$$

Notice that $x_1 \star x_2 = e^\hbar (x_2 \star x_1)$.

We now specify a microlocal polarization on $(X, \mathcal{A}_X)$. For that purpose, we set

(i) $\Sigma = \mathbb{C}^\times$,

(ii) the projection $\pi_1 : X \to \Sigma,\ (x_1, x_2) \mapsto x_1$,

(iii) the Lagrangian immersion $\iota\colon \Sigma \to X,\ x \mapsto (x, 1)$,

(iv) $\mathcal{Q}$ is the quotient of $\mathcal{A}_X$ by the left ideal of $\mathcal{A}_X$ generated by the section $x_2 - 1$.

(v) We endow Σ with the following algebra of operators. Consider the coordinate x on $\mathbb{C}^\times$. Let $\mathcal{S}_{\mathbb{C}^*}$ be the sub-algebra of $\mathcal{D}^\hbar_{\mathbb{C}^*}$ generated by $\mathcal{O}^\hbar_{\mathbb{C}^*}$, $e^{\hbar x\partial_x}$ and $e^{-\hbar x\partial_x}$. We write S for the operator $e^{\hbar x\partial_x}$ and since $e^{\hbar x\partial_x}e^{-\hbar x\partial_x} = e^{-\hbar x\partial_x}e^{\hbar x\partial_x} = \mathrm{id}$, the operator $e^{-\hbar x\partial_x}$ is naturally denoted by S^{-1}. There is a morphism of left $\pi^{-1}\mathcal{O}_{\mathbb{C}^*}$-algebras defined by

$$\psi\colon \pi^{-1}\mathcal{S}_{\mathbb{C}^*} \to \mathcal{A}_X,\ f(x) \ni \mathcal{O}^\hbar_{\mathbb{C}^*} \mapsto f(x_1),\ S^n \mapsto x_2^n \text{ for } n \in \mathbb{Z}.$$

According to [GS12b, GS12a], the quantum curves used to quantize the A-polynomial, a knot invariant, are sections of $\mathcal{S}_{\mathbb{C}^*}$.

Proposition 2.13 ([Pet18c, Proposition 4.20])**.** *The algebra $\mathcal{A}_X$ is flat over $\pi^{-1}\mathcal{S}_{\mathbb{C}^*}$.*

Example 2.14. Consider the symplectic surface $X = (\mathbb{C}^* \times \mathbb{C}, (dx_1 \wedge dx_2)/x_1)$. Here, there are several natural choices of polarizations that correspond respectively to quantum curves encountered in the study of Hurwitz numbers [MSS13] and Gromov-Witten invariants [DBMN+17]. We endow X with the star-algebra $\mathcal{A}_X$ whose star-product is given by:

$$f \star g = \sum_{k \geq 0} \frac{\hbar^k}{k!} \partial^k_{x_2} f\ (x_1\partial_{x_1})^k g.$$

A) We now specify a microlocal polarization on $(X, \mathcal{A}_X)$:

(i) $\Sigma = \mathbb{C}^*$,

(ii) the projection $\pi_1 : \mathbb{C}^* \times \mathbb{C} \to \mathbb{C}^*$, $(x_1, x_2) \mapsto x_1$,

(iii) the Lagrangian immersion $\iota_1 : \Sigma \to X$, $x \mapsto (x, 0)$,

(iv) $\mathcal{Q}$ is the quotient of $\mathcal{A}_X$ by the left ideal of $\mathcal{A}_X$ generated by the section x_2.

(v) We endow Σ, with the Rees algebra of differential operator filtered by the order. Since x is invertible a generator of this algebra is $\hbar x\partial_x$. There is a morphism of $\pi^{-1}\mathcal{O}^\hbar_\Sigma$-algebra defined by

$$\psi\colon \pi^{-1}R(\mathcal{D}_\Sigma) \to \mathcal{A}_X,\ f(x) \ni \mathcal{O}^\hbar_{\mathbb{C}^*} \mapsto f(x_1),\ \hbar x\partial_x \mapsto x_2^n \text{ for } n \in \mathbb{N}.$$

This polarization corresponds to the setting used to study Hurwitz numbers (cf. for instance [MSS13]).

Proposition 2.15. *The algebra $\mathcal{A}_X$ is flat over $\pi^{-1}R(\mathcal{D}_{\mathbb{C}^*})$.*

B) We will now consider a microlocal polarization on $(X, \mathcal{A}_X^{\mathrm{op}})$. Consider the following data.

(i) $\Sigma = \mathbb{C}$,

(ii) the projection $\pi_2 : \mathbb{C}^* \times \mathbb{C} \to \mathbb{C}$, $(x_1, x_2) \mapsto x_2$,

(iii) the Lagrangian immersion is given by $\iota_2 : \mathbb{C} \to X$, $x \mapsto (1, x)$,

(iv) $\mathcal{Q}$ is the quotient of $\mathcal{A}_X^{\mathrm{op}}$ by the left ideal of $\mathcal{A}_X^{\mathrm{op}}$ generated by the section $x_1 - 1$,

(v) Consider Σ with the coordinate x. Let $\mathcal{T}_{\mathbb{C}}$ be the sub-algebra of $\mathcal{D}_{\mathbb{C}}^{\hbar}$ generated by $\mathcal{O}_{\mathbb{C}}$ and $e^{\hbar \partial_x}$ and $e^{-\hbar\partial_x}$. We denote by T the operator $e^{\hbar\partial_x}$ and since $e^{-\hbar\partial_x}$ is the inverse of T, we naturally denote it by T^{-1}. There is a morphism of $\pi^{-1}\mathcal{O}_{\mathbb{C}}^{\hbar}$-modules defined by

$$\psi\colon \pi^{-1}\mathcal{T}_{\mathbb{C}} \to \mathcal{A}_X^{\mathrm{op}},\ f(x) \ni \mathcal{O}_{\mathbb{C}}^{\hbar} \mapsto f(x_2),\ T^n \mapsto x_1^n \text{ for } n \in \mathbb{Z}. \tag{2.3.1}$$

This polarization corresponds to the setting used to study Gromov-Witten invariants in [DBMN+17].

Proposition 2.16 ([Pet18c, Proposition 4.22])**.** *The algebra $\mathcal{A}_X^{\mathrm{op}}$ is flat over $\pi^{-1}\mathcal{T}_{\mathbb{C}}$.*

Remark 2.17. If instead of working analytically one works algebraically, it follows by a direct adaptation of [LS91] that these two polarizations are related via the Mellin transform. Consider the algebra A_{Σ_1} given by the free algebra over $\mathbb{C}$ generated by t, t^{-1}, $\hbar$, $\hbar t\partial_t$ together with the relations

$$[\hbar, t] = 0, \quad [\hbar, \hbar t\partial_t] = 0 \qquad\qquad [\hbar t\partial_t, t] = \hbar t, \quad tt^{-1} = t^{-1}t = 1.$$

Similarly, consider the algebra A_{Σ_2} given by the free algebra over $\mathbb{C}$ generated by s, $\hbar$, $e^{\hbar\partial_s}$, $e^{-\hbar\partial_s}$ together with the relations

$$[\hbar, s] = 0, \quad [\hbar, e^{\hbar\partial_s}] = 0 \qquad\qquad [e^{\hbar\partial_s}, s] = \hbar e^{\hbar\partial_s}, \quad e^{\hbar\partial_s}e^{-\hbar\partial_s} = e^{\hbar\partial_s}e^{-\hbar\partial_s} = 1.$$

We define the following map

$$\phi\colon A_{\Sigma_2} \to A_{\Sigma_1}, \quad \hbar \mapsto \hbar, \quad e^{\hbar\partial_s} \mapsto t, \quad s \mapsto -\hbar t\partial_t.$$

This application is well defined and is an isomorphism of algebras. It provides the Mellin transform by considering an A_{Σ_1}-module as an A_{Σ_2}-modules via the morphism ϕ.

Chapter 3

The codimension-three conjecture

3.1 Introduction

The codimension-three conjecture is an analytic extension problem for microdifferential modules formulated by Masaki Kashiwara at the end of the 70's and solved in 2014 in [KV14] by Masaki Kashiwara and Kari Vilonen. As the theory of DQ-modules can be considered as a generalization of the theory of microdifferential modules to non-necessarily homogenous symplectic manifold, it is natural to try to prove an analogue of this conjecture for DQ-modules. We established such an analogue for holonomic DQ-modules in [Pet18a]. The codimension-three conjecture for holonomic DQ-modules does not relies on the result for microdifferential modules. Nonetheless, our proof follows the general strategy of [KV14]. An important difference is that in [KV14], the author work on the cotangent bundle seen as an *homogeneous* symplectic manifold which is not possible while working with DQ-modules and force us to implement their strategy quite differently. The two results are formally very close which is immediately visible in the two statements below.

Theorem 3.1 ([KV14, Theorem 1.2]). *Let X be a complex manifold, U an open subset of T^*X, Λ a closed Lagrangian analytic subset of U, and Y a closed analytic subset of Λ such that $\operatorname{codim}_\Lambda Y \geq 3$. Let $\mathcal{E}_X$ the sheaf of microdifferential operators on T^*X and $\mathcal{M}$ be a holonomic $(\mathcal{E}_X|_{U\setminus Y})$-module whose support is contained in $\Lambda \setminus Y$. Assume that $\mathcal{M}$ possesses an $(\mathcal{E}_X(0)|_{U\setminus Y})$-lattice. Then $\mathcal{M}$ extends uniquely to a holonomic module defined on U whose support is contained in Λ.*

Theorem 3.2 ([Pet18a, Theorem 1.6]). *Let X be a complex manifold endowed with a DQ-algebroid stack $\mathcal{A}_X$ such that the associated Poisson structure is symplectic. Let Λ be a closed Lagrangian analytic subset of X and Y a closed analytic subset of Λ such that $\operatorname{codim}_\Lambda Y \geq 3$. Let $\mathcal{M}$ be a holonomic $(\mathcal{A}_X^{\mathrm{loc}}|_{X\setminus Y})$-module, whose support is contained in $\Lambda \setminus Y$. Assume that $\mathcal{M}$ has an $\mathcal{A}_X|_{X\setminus Y}$-lattice. Then $\mathcal{M}$ extends uniquely to a holonomic module defined on X whose support is contained in Λ.*

Hence, it is natural to study the relation between these two results. The codimension-three conjecture for formal holonomic microdifferential modules does not implies the one for holonomic DQ-modules. This can be seen by using the counter-example [DK11, Proposition A.2.3] due to M. Kashiwara and P. Schapira which shows that there are holonomic DQ-modules which are not associated with holonomic microdifferential modules. On the contrary, the original codimension-three conjecture for formal microdifferential modules can be deduced from the codimension-three result for holonomic DQ-modules by using the result of [Pet18b] which by further developing the theory of *holomorphic Frobenius actions* introduced by M. Kashiwara and R. Rouquier in [KR08] allows to precise the relation between DQ-modules and microdifferential modules.

The various versions of the codimension-three conjecture are bearing striking similarities with several analytic extensions results. For instance, the codimension-three conjecture is a microlocal analogue of the following result due to Frisch-Guenot [FG69], Trautmann [Tra67] and Siu [Siu69].

Theorem 3.3 (Frisch-Guenot, Trautmann, and Siu). *Let X be a complex manifold, let S be a closed analytic subset of X and $j : X \setminus S \to X$ be the open embedding of $X \setminus S$ into X. If $\mathcal{F}$ is a reflexive coherent $\mathcal{O}_{X\setminus S}$-module and* $\operatorname{codim} S \geq 3$ *then $j_*\mathcal{F}$ is a coherent $\mathcal{O}_X$-module.*

The above result answered a question of J. P. Serre [Ser66]. Building upon this result M. Kashiwara and K. Vilonen extended Theorem 3.3 to reflexive coherent sheaves over $\mathcal{O}_X^\hbar$.

Theorem 3.4 ([KV14, Theorem 1.6]). *Let X be a complex manifold, let Y be a closed analytic subset of X and $j : X \setminus Y \to X$ the open embedding of $X \setminus Y$ into X. If $\mathcal{N}$ is a coherent reflexive $\mathcal{O}_{X\setminus Y}^\hbar$-module and* $\operatorname{codim} Y \geq 3$ *then $j_*\mathcal{N}$ is a coherent $\mathcal{O}_X^\hbar$-module.*

This theorem which is a cornerstone of the proofs of the codimension-three conjecture for holonomic microdifferential modules as well as for holonomic DQ-modules can be considered as a codimension-three type result for reflexive DQ-modules over the trivial DQ-algebra.

These examples suggest that it may be possible to unify and generalize codimension-three type statements to arbitrary Poisson manifolds. We mention here a few conjectural generalizations. The main difference between these generalizations is that they are either concerned with $\mathcal{A}_X$ or $\mathcal{A}_X^{\mathrm{loc}}$-modules. This impacts how the notion of reflexivity/holonomicity is generalized.

Here, is a first possibility where the notion of reflexivity is generalized as follow.

Definition 3.5. Let $(X, \mathcal{A}_X)$ be a complex manifold endowed with a DQ-algebra $\mathcal{A}_X$, $\mathcal{M}$ a coherent $\mathcal{A}_X$-module and $d \in \mathbb{N}$. We say that $\mathcal{M}$ is d-reflexive if

(i) $\mathcal{M}$ has no $\hbar$-torsion,

(ii) $\mathcal{E}xt^j_{\mathcal{A}_X}(\mathcal{M}, \mathcal{A}_X) = 0$ for $j < d$,

(iii) $\mathcal{M} \xrightarrow{\sim} \mathcal{E}xt^d_{\mathcal{A}_X}(\mathcal{E}xt^d_{\mathcal{A}_X}(\mathcal{M}, \mathcal{A}_X), \mathcal{A}_X)$.

When $d = 0$ and $\mathcal{A}_X = \mathcal{O}^\hbar_X$ the conjecture below is Theorem 3.4.

Conjecture. Let $(X, \mathcal{A}_X)$ be a complex manifold endowed with a DQ-algebra $\mathcal{A}_X$. Let Y be a closed complex analytic subset of X such that $\operatorname{codim}_X Y \geq d + 3$ and denote by $j\colon X \setminus Y \hookrightarrow X$ the open inclusion. Let $\mathcal{M}$ be a coherent $\mathcal{A}_X|_{X\setminus Y}$-module. If $\mathcal{M}$ is d-reflexive, then $j_*\mathcal{M}$ is a coherent $\mathcal{A}_X$-module.

A second possibility is arising from the homological characterization of holonomicity that we recall below.

Proposition 3.6 ([KS12, Lemma 7.2.2])**.** *Let $\mathcal{M}$ be a holonomic $\mathcal{A}^{\mathrm{loc}}_X$-module. Then, the object $\mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{A}^{\mathrm{loc}}_X}(\mathcal{M}, \mathcal{A}^{\mathrm{loc}}_X)[d_X/2]$ is concentrated in degree zero and is holonomic.*

We generalize the reflexivity condition in the direction of holonomicity as shown by the following result.

Conjecture. Let $(X, \mathcal{A}_X)$ be a complex manifold endowed with a DQ-algebra $\mathcal{A}_X$. Let Y be a closed complex analytic subset of X such that $\operatorname{codim}_X Y \geq d + 3$ and denote by $j\colon X \setminus Y \hookrightarrow X$ the open inclusion. Let $\mathcal{M}$ be a coherent $\mathcal{A}^{\mathrm{loc}}_X|_{X\setminus Y}$-module. Assume that $\mathcal{M}$ has an $\mathcal{A}_{X\setminus Y}$-lattice and that $\mathcal{E}xt^j_{\mathcal{A}^{\mathrm{loc}}_X}(\mathcal{M}, \mathcal{A}^{\mathrm{loc}}_X) = 0$ for $j \neq d$. Then $j_*\mathcal{M}$ is a coherent $\mathcal{A}^{\mathrm{loc}}_X$-module.

The above conjecture is the codimension-three for holonomic DQ-modules when X is symplectic and $d = d_X/2$ (this is Theorem 3.2). Our proof of the codimension-three conjecture for holonomic DQ-modules do not generalize directly to this setting as it relies crucially on the fact that, locally, on a complex symplectic manifold there is up to isomorphism a unique star-algebra quantizing this symplectic manifold.

3.2 The codimension-three conjecture for holonomic DQ-modules

The strategy of the proof is adapted from [KV14]. We keep the notation of Theorem 3.2. If $\mathcal{M}$ is a DQ-modules satistfying the assumption of this theorem and $j\colon X \setminus Y \hookrightarrow X$ is the open inclusion, the strategy of the proof is to establish the coherency of $j_*\mathcal{M}$ by reducing the problem to a commutative setting where Theorem 3.4 can be applied.

We briefly describe each of the majors steps of the proof.

First, we establish that if a holonomic extension $\mathcal{M}'$, to X, of the holonomic $\mathcal{A}^{\mathrm{loc}}_X|_{X\setminus Y}$-module $\mathcal{M}$ exists then it is isomorphic to $j_*\mathcal{M}$. Moreover, the Remmert-Stein Theorem implies that the support of $j_*\mathcal{M}$ is a Lagrangian subvariety.

Then, we are left to prove the coherency of $j_*\mathcal{M}$. We will rely on the following coherence criterion which is a non-commutative version of a folklore result closely related to the results in [Hou61]. We start by a few definitions.

Let M be an open subset of $\mathbb{C}^n$ endowed with the coordinates $(x) = (x_1, \dots, x_n)$. We set $X = T^*M$ and denote by (x, u) the associated symplectic coordinate system on X. We let

$$\rho : T^*M \to M$$

be the projection and designate by $\mathcal{A}_X$ the Wick star-algebra on X. We denote by $\mathcal{B}_M$ the sheaf on M defined as follows. It is the subalgebra of $\mathcal{A}_X|_M$ generated by the algebra $\mathcal{O}^\hbar_M$ and the u_i's, $1 \leq i \leq n$. Hence, as a sheaf over $\mathbb{C}^\hbar_M$, $\mathcal{B}_M \simeq \mathcal{O}^\hbar_M \otimes \mathbb{C}[u_1, \cdots, u_n]$. We denote as usual by $\mathcal{B}^{\mathrm{loc}}_M$ the $\hbar$-localization of $\mathcal{B}_M$, that is, $\mathcal{B}^{\mathrm{loc}}_M := \mathbb{C}^{\hbar,\mathrm{loc}}_M \underset{\mathbb{C}^\hbar_M}{\otimes} \mathcal{B}_M$. We also note that $\mathcal{B}_M/\hbar\mathcal{B}_M \simeq \mathcal{O}_M[u]$ where $\mathcal{O}_M[u]$ is a shorthand for $\mathcal{O}_M[u_1, \dots, u_n]$.

Definition 3.7. (i) We denote by $\mathrm{Mod}_{\mathcal{O}^\hbar_M-\mathrm{coh}}(\mathcal{B}_M)$ the abelian full subcategory of the category $\mathrm{Mod}(\mathcal{B}_M)$ consisting of modules which are $\mathcal{O}^\hbar_M$-coherent.

(ii) Let $\mathcal{N} \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{B}^{\mathrm{loc}}_M)$ and $\mathcal{L}$ a $\mathcal{B}_M$-submodule of $\mathcal{N}$. We say that $\mathcal{L}$ is a finite $\mathcal{B}_M$-lattice of $\mathcal{N}$ if $\mathcal{L} \in \mathrm{Mod}_{\mathcal{O}^\hbar_M-\mathrm{coh}}(\mathcal{B}_M)$ and $\mathcal{B}^{\mathrm{loc}}_M \underset{\mathcal{B}_M}{\otimes} \mathcal{L} \simeq \mathcal{N}$.

(iii) We denote by $\mathrm{Mod}_{\mathrm{lfl}}(\mathcal{B}^{\mathrm{loc}}_M)$ the abelian full subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{B}^{\mathrm{loc}}_M)$ consisting of modules admitting locally a finite $\mathcal{B}_M$-lattice.

Definition 3.8. (i) The module $\mathcal{M} \in \mathrm{Mod}(\mathcal{A}^{\mathrm{loc}}_X)$ is ρ-finite if the restriction of ρ to the support of $\mathcal{M}$ is a finite morphism. We denote by $\mathrm{Mod}_{\rho-\mathrm{fin}}(\mathcal{A}^{\mathrm{loc}}_X)$ the full abelian sub-category of $\mathrm{Mod}(\mathcal{A}^{\mathrm{loc}}_X)$ consisting of ρ-finite modules.

(ii) We denote by $\mathrm{Mod}_{\rho-\mathrm{coh}}(\mathcal{A}^{\mathrm{loc}}_X)$ the full abelian sub-category of $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{A}^{\mathrm{loc}}_X)$ consisting of ρ-finite modules.

Remark 3.9. Gabber's Theorem and the ρ-finiteness condition implies that a coherent ρ-finite $\mathcal{A}^{\mathrm{loc}}_X$-module $\mathcal{N}$ is holonomic.

We define the functor

$$\rho^\natural : \mathrm{Mod}(\mathcal{B}^{\mathrm{loc}}_M) \to \mathrm{Mod}(\mathcal{A}^{\mathrm{loc}}_X), \quad \rho^\natural\mathcal{N} :\, = \mathcal{A}^{\mathrm{loc}}_X \underset{\rho^{-1}\mathcal{B}^{\mathrm{loc}}_M}{\otimes} \rho^{-1}\mathcal{N}$$

and the functor

$$\rho_\natural : \mathrm{Mod}(\mathcal{A}^{\mathrm{loc}}_X) \to \mathrm{Mod}(\mathcal{B}^{\mathrm{loc}}_M),$$

which is defined as the composition of the direct image functor ρ_* with the forgetful functor $\mathrm{Mod}(\rho_*\mathcal{A}^{\mathrm{loc}}_X) \to \mathrm{Mod}(\mathcal{B}^{\mathrm{loc}}_M)$. Hence we get the following adjunction

$$\mathrm{Mod}(\mathcal{A}_X^{\mathrm{loc}}) \underset{\rho^{\natural}}{\overset{\rho_{\natural}}{\rightleftarrows}} \mathrm{Mod}(\mathcal{B}_M^{\mathrm{loc}})$$

for which, we obtain the following result proved in [Pet18a, Theorem 3.27].

Theorem 3.10. *The functors*

$$\mathrm{Mod}_{\rho-\mathrm{coh}}(\mathcal{A}_X^{\mathrm{loc}}) \underset{\rho^{\natural}}{\overset{\rho_{\natural}}{\rightleftarrows}} \mathrm{Mod}_{\mathrm{lfl}}(\mathcal{B}_M^{\mathrm{loc}})$$

are well-defined and are equivalences of categories inverse to each other. Moreover, if $\mathcal{M} \in \mathrm{Mod}_{\rho-\mathrm{fin}}(\mathcal{A}_X^{\mathrm{loc}})$ *and* $\rho_{\natural}\mathcal{M} \in \mathrm{Mod}_{\mathrm{lfl}}(\mathcal{B}_M^{\mathrm{loc}})$*, then* $\mathcal{M} \in \mathrm{Mod}_{\rho-\mathrm{coh}}(\mathcal{A}_X^{\mathrm{loc}})$*.*

Now our aim is to show that we can always (locally) reduce the problem to a situation where we can apply the coherency criterion of Theorem 3.10. Hence, we can assume that we are in an open neighborhood of a point $p \in Y$ and using Darboux theorem, we can furthermore assume that U is an open subset of a symplectic vector space. In particular, we established in [Pet18a, Theorem 5.6].

Theorem 3.11. *Let* (E, ω) *be a complex symplectic vector space of finite dimension, let* U *be an open subset of* E*, let* Λ *be a closed Lagrangian analytic subset of* U *and let* $p \in \Lambda$*. There exist a linear Lagrangian subspace* λ *of* E*, an open neighborhood* V *of* p *and an open neighborhood* M *of* $\pi_\lambda(p)$ *the image of* p *by the canonical projection* $\pi_\lambda : E \to E/\lambda$*, such that the map*

$$\rho := \pi_\lambda|_V^M : V \to M$$

is finite when restricted to $\Lambda \cap V$*.*

Relying on the above statement and shrinking U if necessary, we obtain a map $\rho\colon U \to M$ such that $M = \rho(U)$, ρ restricted to Λ is finite and setting $N = \rho(Y)$, we can further assume that

$$\rho|_Y \colon Y \to N$$

is an isomorphism and that $\operatorname{codim} N \geq 3$. Moreover, we can find a symplectic coordinate system (x, u) such that $\rho(x, u) = x$ and as on a symplectic manifold all star-algebra are locally isomorphic, we can assume that the star-algebra quantizing U is the Wick star algebra. We set $Y' = \rho^{-1}(N)$ and let $i\colon U \setminus Y' \to U \setminus Y$ be the inclusion. We also consider the following restriction of ρ

$$\rho'\colon U \setminus Y' \to M \setminus N$$

and obtain the following cartesian square

$$\begin{array}{ccc} U\setminus\rho^{-1}(N) & \xhookrightarrow{j\circ i} & U \\ {\scriptstyle\rho'}\downarrow & \square & \downarrow{\scriptstyle\rho} \\ M\setminus N & \xhookrightarrow[j']{} & M. \end{array}$$

Using the unicity of the extension under the hypothesis of the codimension-three conjecture, one shows that

$$j_*i_*i^{-1}\mathcal{M}\simeq j_*\mathcal{M}$$

Hence, we are reduced to prove the coherency of $j_*i_*i^{-1}\mathcal{M}$.

The following proposition allows us to replace the lattice $\mathcal{N}$ of $\mathcal{M}$ by the lattice $\mathcal{N}'=\mathcal{E}xt^n_{\mathcal{A}_X}(\mathcal{E}xt^n_{\mathcal{A}_X}(\mathcal{N},\mathcal{A}_X),\mathcal{A}_X)$.

Proposition 3.12 ([Pet18a, Proposition 2.22])**.** *Let $\mathcal{M}$ be a holonomic $\mathcal{A}_X^{\mathrm{loc}}$-module. Assume that it has a lattice $\mathcal{N}$. Then $\mathcal{E}xt^n_{\mathcal{A}_X}(\mathcal{E}xt^n_{\mathcal{A}_X}(\mathcal{N},\mathcal{A}_X),\mathcal{A}_X)$ is also a lattice of $\mathcal{M}$.*

Now, a duality argument, shows that $\rho'_*i^{-1}\mathcal{N}'$ is the dual of a $\mathcal{O}^{\hbar}_{M\setminus N}$-module. Thus, it is a coherent reflexive $\mathcal{O}^{\hbar}_{M\setminus N}$-module. Hence, applying Theorem 3.4, we get that $j'_*\rho'_*i^{-1}\mathcal{N}'$ is a coherent $\mathcal{O}^{\hbar}_M$-module. Using again the reflexivity of $\rho'_*i^{-1}\mathcal{N}'$, one gets by [Pet18a, Proposition 6.3] that

$$(j'_*\rho'_*i^{-1}\mathcal{N}')^{\mathrm{loc}}\simeq j'_*\rho'_*i^{-1}\mathcal{M}. \tag{3.2.1}$$

Thus, the $\mathcal{B}_M$-module $j'_*\rho'_*i^{-1}\mathcal{N}'$ is a $\mathcal{O}^{\hbar}_M$-lattice of the $\mathcal{B}^{\mathrm{loc}}_M$-module $j'_*\rho'_*i^{-1}\mathcal{M}$. This together with the below commutative diagram

$$\begin{array}{ccc} \rho_*(j\circ i)_*i^{-1}\mathcal{M} & \xrightarrow{\sim} & j'_*\rho'_*i^{-1}\mathcal{M} \\ \uparrow & & \uparrow \\ \rho_*(j\circ i)_*i^{-1}\mathcal{N}' & \xrightarrow{\sim} & j'_*\rho'_*i^{-1}\mathcal{N}' \end{array} \tag{3.2.2}$$

implies that the $\mathcal{B}_M$-module $\rho_*(j\circ i)_*i^{-1}\mathcal{N}'$ is a $\mathcal{O}^{\hbar}_M$-lattice of the $\mathcal{B}^{\mathrm{loc}}_M$-module $\rho_*(j\circ i)_*i^{-1}\mathcal{M}$. It follows from Theorem 3.10 that the module $(j\circ i)_*i^{-1}\mathcal{M}$ is a coherent $\mathcal{A}^{\mathrm{loc}}_M$-modules. Hence, we have proven that $j_*\mathcal{M}$ is an holonomic module.

3.3 From DQ-modules to microdifferential modules

The statement of the codimension-three conjecture for holonomic DQ-modules and microdifferential modules are very similar. Moreover, the DQ-modules version seems to be more general than the microdifferential one. Hence, it is natural to try to deduce the microdifferential version from the DQ one.

It is possible to associate to any microdifferential module a DQ-module. Indeed, let M be a complex manifold and X be an open subset of T^*M, then using the morphism of algebras appearing in Diagram (2.2.2), we get the functor of extension of scalar

$$\mathrm{Mod}(\widehat{\mathcal{E}}_X) \to \mathrm{Mod}(\widehat{\mathcal{W}}_X),\ \mathcal{M} \mapsto \widehat{\mathcal{W}}_X \underset{\widehat{\mathcal{E}}_X}{\otimes} \mathcal{M}.$$

Hence, if we are able to identify the DQ-modules which are obtain by extension of scalar of a microdifferential module and reconstruct this microdiferential module, we are able to recover the codimension-three conjecture for formal microdifferential modules. As $\widehat{\mathcal{E}}_X$ quantize the cotangent bundle as a homogeneous symplectic manifold, it is necessary to endow $\widehat{\mathcal{W}}_X$-module with an extra structure encoding the compatibility of the DQ-modules with the action of $\mathbb{C}^\times$ on the cotangent bundle. Such a thing can be achieved through the use of the notion of *Holomorphic Frobenius action* or F-actions introduced by M. Kashiwara and R. Rouqier in [KR08]. They were initially introduced to formulate an analogue of the Beilinson-Bernstein localization for rational Cherednik algebras.

Roughly speaking, given a Poisson manifold endowed with an holomorphic action of $\mathbb{C}^\times$ and a DQ-algebra, an holomorphic F-action is an action of $\mathbb{C}^\times$ on the DQ-algebra compatible with the action on the manifold and acting with a weight on $\hbar$. The non-trivial action on the deformation parameter allow to rescale the DQ-algebra and equivariant DQ-modules with respects to the deformation parameter $\hbar$. Hence, providing a way to pass from object defined over $\mathbb{C}((\hbar))$ to objects defined over $\mathbb{C}$.

The original definition [KR08] of an F-action is a punctual definition. This makes it difficult to use for problems of global nature as for instance extending an F-action through an analytic subset, a situation encountered in tackling the codimension-three conjecture. In [Pet18b], we provided a functorial definition of this type of action. To obtain a notion suited to our need, we gave a reformulation inspired from the notion of G-linearization of a coherent sheaf [FKM02, Ch. 1 §3]. Using this definition, we established an equivalence between the category of coherent DQ-modules endowed with an F-action and the category of coherent modules over the algebra of invariant sections. When specializing the equivalence to the case where the DQ-algebra is $\widehat{\mathcal{W}}_X$, one obtains an equivalence of categories between F-equivariant coherent-$\widehat{\mathcal{W}}_X$ modules and formal microdifferential modules. This equivalence allows to recover the codimension-three conjecture for formal microdiferential modules from the one for DQ-modules.

3.3.1 Holomorphic Frobenius action

In this section, we introduce our formulation of F-actions. We start by setting a few notations.

Let $(X, \{\cdot\,,\,\cdot\})$ be a complex Poisson manifold. We assume that it comes equipped with a torus action, $\mathbb{C}^\times \to \mathrm{Aut}(X),\ t \mapsto \mu_t$ such that $\mu_t^*\{f,\, g\} = t^{-m}\{\mu_t^* f,\, \mu_t^* g\}$ with $m \in \mathbb{Z}^*$.

Notations 3.13.

- We denote by $\sigma\colon \mathbb{C}^\times \times \mathbb{C}^\times \to \mathbb{C}^\times$ the group law of $\mathbb{C}^\times$,

- $\mu\colon \mathbb{C}^\times \times X \to X$ the action of $\mathbb{C}^\times$ on X,
- $\widetilde{\mu}\colon \mathbb{C}^\times \times X \to \mathbb{C}^\times \times X,\ (t,x) \mapsto (t, \mu(t,x))$,
- for $t \in \mathbb{C}^\times$, the morphism

$$i_t\colon X \to \mathbb{C}^\times \times X,\ x \mapsto (t,x).$$

- We write $\widetilde{\mu}_t$ for the composition $\widetilde{\mu} \circ i_t$ (Note that $\mu_t = \mu \circ i_t$).
- Consider the product of manifolds $\mathbb{C}^\times \times X$. We denote by p_i the i-th projection.
- Consider the product of manifolds $\mathbb{C}^\times \times \mathbb{C}^\times \times X$. We denote by q_i the i-th projection, and by q_{ij} the (i,j)-th projection (*e.g.*, q_{13} is the projection from $\mathbb{C}^\times \times \mathbb{C}^\times \times X$ to $\mathbb{C}^\times \times X$, $(t_1, t_2, x_3) \mapsto (t_1, x_3)$).
- We write $a_X\colon X \to \mathrm{pt}$ for the unique map from X to the point.
- Throughout $\mathcal{A}_X$ is DQ-algebra and we write $\mathcal{A}_{\mathbb{C}^\times \times X}$ for the DQ-algebra $\mathcal{O}^\hbar_{\mathbb{C}^\times} \boxtimes \mathcal{A}_X$.

As F-action are defined in the complex analytic setting, we need the following extension lemma to define them precisely.

Lemma 3.14 ([Pet18b, Lemma 4.2]). *Let $\widetilde{\theta} : \widetilde{\mu}^{-1}\mathcal{A}_{\mathbb{C}^\times \times X} \to \mathcal{A}_{\mathbb{C}^\times \times X}$ be a morphism of sheaves of $p_1^{-1}\mathcal{O}_{\mathbb{C}^\times}$-algebras such that the adjoint morphism $\psi : \mathcal{A}_{\mathbb{C}^\times \times X} \to \widetilde{\mu}_* \mathcal{A}_{\mathbb{C}^\times \times X}$ is a continuous morphism of Fréchet $\mathbb{C}$-algebras. Then the dashed arrow, in the diagram below, is filled by a unique morphism $\widetilde{\lambda}$ of $q_{12}^{-1}\mathcal{O}_{\mathbb{C}^\times \times \mathbb{C}^\times}$-algebras. If $\widetilde{\theta}$ is an isomorphism then $\widetilde{\lambda}$ is also an isomorphism.*

$$\begin{array}{ccc}
(\mathrm{id}_{\mathbb{C}^\times} \times \widetilde{\mu})^{-1}(\mathcal{O}_{\mathbb{C}^\times} \boxtimes \mathcal{A}_{\mathbb{C}^\times \times X}) & \xrightarrow{\mathrm{id} \times \widetilde{\theta}} & \mathcal{O}_{\mathbb{C}^\times} \boxtimes \mathcal{A}_{\mathbb{C}^\times \times X} \\
\downarrow & & \downarrow \\
(\mathrm{id}_{\mathbb{C}^\times} \times \widetilde{\mu})^{-1}\mathcal{A}_{\mathbb{C}^\times \times \mathbb{C}^\times \times X} & \underset{\widetilde{\lambda}}{\dashrightarrow} & \mathcal{A}_{\mathbb{C}^\times \times \mathbb{C}^\times \times X}
\end{array}$$

Before being able to define F-actions on DQ-algebras, we need to define several morphisms. It follows from the definition of the external product of DQ-algebras that there is a canonical morphism

$$q_{23}^\sharp\colon q_{23}^{-1}\mathcal{A}_{\mathbb{C}^\times \times X} \to \mathcal{A}_{\mathbb{C}^\times \times \mathbb{C}^\times \times X}.$$

Using the same notations an in Lemma 3.14, we define the morphism λ as the composition

$$\lambda\colon (\mathrm{id} \times \mu)^{-1}\mathcal{A}_{\mathbb{C}^\times \times X} \xrightarrow{(\mathrm{id} \times \widetilde{\mu})^{-1} q_{23}^\sharp} (\mathrm{id} \times \widetilde{\mu})^{-1}\mathcal{A}_{\mathbb{C}^\times \times \mathbb{C}^\times \times X} \xrightarrow{\widetilde{\lambda}} \mathcal{A}_{\mathbb{C}^\times \times \mathbb{C}^\times \times X}. \tag{3.3.1}$$

We define the evaluation functor at $t \in \mathbb{C}^\times$ as follows:

$$\begin{aligned} \mathrm{Ev_t}\colon \mathrm{Mod}(p_1^{-1}\mathcal{O}_{\mathbb{C}^\times}) &\to \mathrm{Mod}(\mathbb{C}_X) \\ \mathcal{M} &\mapsto a_X^{-1}(\mathcal{O}_{\mathbb{C}^\times,t}/\mathfrak{m}_t) \underset{a_X^{-1}\mathcal{O}_{\mathbb{C}^\times,t}}{\otimes} i_t^{-1}\mathcal{M} \simeq i_t^{-1}\mathcal{M}/a_X^{-1}\mathfrak{m}_t\, i_t^{-1}\mathcal{M}. \end{aligned}$$

In particular, $\mathrm{Ev}_t(\mathcal{A}_{\mathbb{C}^\times \times X}) \simeq \mathcal{A}_X$ and $\mathrm{Ev}_t(\widetilde{\mu}^{-1}\mathcal{A}_{\mathbb{C}^\times \times X}) \simeq \mu_t^{-1}\mathcal{A}_X$.

We are now able to define F-actions on DQ-algebras.

Definition 3.15 ([Pet18b, Definition 4.3])**.** An F-action with exponent m on $\mathcal{A}_X$ is the data of an isomorphism of $p_1^{-1}\mathcal{O}_{\mathbb{C}^\times}$-algebras $\widetilde{\theta}\colon \widetilde{\mu}^{-1}\mathcal{A}_{\mathbb{C}^\times \times X} \to \mathcal{A}_{\mathbb{C}^\times \times X}$ such that

(a) the morphism $\theta_t := \mathrm{Ev_t}(\widetilde{\theta})$ satisfies $\theta_1 = \mathrm{id}$,

(b) for every $t \in \mathbb{C}^\times$, $\theta_t(\hbar^n) = t^{mn}\hbar^n$,

(c) the adjoint morphism of $\widetilde{\theta}$, $\widetilde{\psi} : \mathcal{A}_{\mathbb{C}^\times \times X} \to \widetilde{\mu}_*\mathcal{A}_{\mathbb{C}^\times \times X}$ is a continuous morphism of Fréchet $\mathbb{C}$-algebras,

(d) setting

$$\theta\colon \mu^{-1}\mathcal{A}_X \xrightarrow{\widetilde{\mu}^{-1}p_2^\sharp} \widetilde{\mu}^{-1}\mathcal{A}_{\mathbb{C}^\times \times X} \xrightarrow{\widetilde{\theta}} \mathcal{A}_{\mathbb{C}^\times \times X}$$

the below diagram commutes,

$$\begin{array}{ccc} (\mathrm{id}_{\mathbb{C}^\times} \times \mu)^{-1}\mu^{-1}\mathcal{A}_X & \xrightarrow{(\mathrm{id}_{\mathbb{C}^\times} \times \mu)^{-1}\theta} & (\mathrm{id}_{\mathbb{C}^\times} \times \mu)^{-1}\mathcal{A}_{\mathbb{C}^\times \times X} \\ \| & & \downarrow \lambda \\ \| & & \mathcal{A}_{\mathbb{C}^\times \times \mathbb{C}^\times \times X} \\ \| & & \uparrow \\ (\sigma \times \mathrm{id}_X)^{-1}\mu^{-1}\mathcal{A}_X & \xrightarrow{(\sigma \times \mathrm{id}_X)^{-1}\theta} & (\sigma \times \mathrm{id}_X)^{-1}\mathcal{A}_{\mathbb{C}^\times \times X} \end{array}$$

where λ is the morphism (3.3.1).

Definition 3.16. An F-action on $\mathcal{A}_X^{\mathrm{loc}}$ is the localization with respect to $\hbar$ of an F-action on $\mathcal{A}_X$.

The data of an F-action on a DQ-algebra provides a morphism of $\mathbb{C}$-ringed spaces.

$$(\mu, \theta)\colon (\mathbb{C}^\times \times X, \mathcal{A}_{\mathbb{C}^\times \times X}^{\mathrm{loc}}) \to (X, \mathcal{A}_X^{\mathrm{loc}}). \tag{3.3.2}$$

It is important to remark that the morphism (3.3.2) is not $\mathbb{C}^\hbar$-linear. The morphism of sheaves

$$\lambda\colon (\mathrm{id} \times \mu)^{-1}\mathcal{A}_{\mathbb{C}^\times \times X}^{\mathrm{loc}} \to \mathcal{A}_{\mathbb{C}^\times \times \mathbb{C}^\times \times X}^{\mathrm{loc}}$$

provides the following morphism of ringed spaces

$$(\mathrm{id} \times \mu, \lambda) \colon (\mathbb{C}^{\times} \times \mathbb{C}^{\times} \times X, \mathcal{A}^{\mathrm{loc}}_{\mathbb{C}^{\times} \times \mathbb{C}^{\times} \times X}) \to (\mathbb{C}^{\times} \times X, \mathcal{A}^{\mathrm{loc}}_{\mathbb{C}^{\times} \times X})$$

while the morphism of sheaves induced by the group law on $\mathbb{C}^{\times}$

$$\sigma^{\sharp} \colon \sigma^{-1}\mathcal{O}_{\mathbb{C}^{\times}} \to \mathcal{O}_{\mathbb{C}^{\times} \times \mathbb{C}^{\times}}$$

induces the following morphism of sheaves

$$\alpha \colon (\sigma \times \mathrm{id}_X)^{-1}\mathcal{A}_{\mathbb{C}^{\times} \times X} \overset{\sigma^{\sharp}\widehat{\otimes}\mathrm{id}}{\longrightarrow} \mathcal{A}_{\mathbb{C}^{\times} \times \mathbb{C}^{\times} \times X}$$

giving rise to the following morphism of ringed spaces

$$(\sigma \times \mathrm{id}_X, \alpha) \colon (\mathbb{C}^{\times} \times \mathbb{C}^{\times} \times X, \mathcal{A}_{\mathbb{C}^{\times} \times \mathbb{C}^{\times} \times X}) \to (\mathbb{C}^{\times} \times X, \mathcal{A}_{\mathbb{C}^{\times} \times X}).$$

The following lemma ensures that the notion of F-action is well-behaved.

Lemma 3.17. *The morphism of sheaves of algebras θ, λ and α are flat.*

We can now state the definition of an F-action on a DQ-module which is an adaptation to the setting of DQ-modules of the notion G-linearization of coherent sheaves found in [FKM02, Ch.1 §3 Definition 1.6].

Definition 3.18. An F-action on a $\mathcal{A}^{\mathrm{loc}}_X$-module $\mathcal{M}$ is the data of an isomorphism of $\mathcal{A}^{\mathrm{loc}}_{\mathbb{C}^{\times} \times X}$-modules

$$\phi \colon \mu^*\mathcal{M} \xrightarrow{\sim} p_2^*\mathcal{M}$$

such that the diagram

$$\begin{array}{ccccccc}
(\mathrm{id}_{\mathbb{C}^{\times}} \times \mu)^*\mu^*\mathcal{M} & \xrightarrow{(\mathrm{id}_{\mathbb{C}^{\times}} \times \mu)^*\phi} & (\mathrm{id}_{\mathbb{C}^{\times}} \times \mu)^* p_2^*\mathcal{M} & \xrightarrow{\sim} & q_{23}^*\mu^*\mathcal{M} & \xrightarrow{q_{23}^*\phi} & q_{23}^* p_2^*\mathcal{M} \\
\| & & & & & & \downarrow\wr \\
\| & & & & & & q_3^*\mathcal{M} \\
\| & & & & & & \uparrow\wr \\
(\sigma \times \mathrm{id}_X)^*\mu^*\mathcal{M} & & & \xrightarrow{(\sigma \times \mathrm{id}_X)^*\phi} & & & (\sigma \times \mathrm{id}_X)^* p_2^*\mathcal{M}
\end{array}$$

commutes.

We write $\mathrm{Mod}_{\mathrm{F}}(\mathcal{A}^{\mathrm{loc}}_X)$ for the category of $(\mathcal{A}^{\mathrm{loc}}_X, \theta)$-modules whose morphisms are the morphisms of $\mathcal{A}^{\mathrm{loc}}_X$-modules compatible with the action of $\mathbb{C}^{\times}$ and $\mathrm{Mod}_{\mathrm{F,coh}}(\mathcal{A}^{\mathrm{loc}}_X)$ for the full subcategory of $\mathrm{Mod}_{\mathrm{F}}(\mathcal{A}^{\mathrm{loc}}_X)$ the objects of which are coherent modules in $\mathrm{Mod}(\mathcal{A}^{\mathrm{loc}}_X)$. We emphasize the we do not require our modules to be coherent in some equivariant sense.

Remark 3.19. Note that our definition of an F-action implies the properties of an F-action stated in [KR08]. Moreover, the classical examples of F-action in the sense of

Kashiwara-Rouquier are F-actions in our sense.

Before stating our first main result concerning F-action, we need one more definition. We say that an open subset U of X is equivariant if $U = \mu(\mathbb{C}^\times \times U)$.

Definition 3.20. Let $(\mathcal{M}, \phi) \in \mathrm{Mod}_{\mathrm{F}}(\mathcal{A}_X^{\mathrm{loc}})$, $U \subset X$ and $s \in \mathcal{M}(U)$.

(i) The section s is locally invariant at x' if there exists an open neighborhood $V \times U' \subset \mathbb{C}^\times \times U$ of $(1, x')$ such that for every $(t, x) \in V \times U'$, $\mu(t, x) \in U$ and $\phi_t(s_{\mu(t,x)}) = s_x$.

(ii) The section s is locally invariant on U if it is locally invariant at every $x' \in U$.

(iii) Assume that $U \subset X$ is equivariant. A section $s \in \mathcal{M}(U)$ is invariant if for every $t \in \mathbb{C}^\times$, $\phi_t(s) = s$.

The following result is the technical heart of [Pet18b]. It is of central importance since it essentialy implies that a coherent DQ-modules endowed with an F-action is equivariantly coherent.

Theorem 3.21 ([Pet18b, Theorem 5.6])**.** *Assume that the action μ is free and let $\mathcal{M} \in \mathrm{Mod}_{\mathrm{F,coh}}(\mathcal{A}_X^{\mathrm{loc}})$. Then $\mathcal{M}$ is locally generated by finitely many locally invariant sections.*

As in the case of action of complex Lie groups, one of the main tools to study F-actions is the derivation canonically associated to such an action. Using this derivation, it is possible to reduce the proof of the above statement to the study of a recursive linear system of first order PDE. The principal difficulty encountered is to ensure that the domain of definition of the solutions obtained at each step of the recusion via the Cauchy-Kowalevski Theorem are defined on a common open set which do not depend of the step. This is achieved by using [BS72].

We can sharpen the above result when the action is free and proper. The following proposition is a consequence of the more precise [Pet18b, Corollary 5.9].

Proposition 3.22. *Assume that the action μ is free and proper. Let $\mathcal{M} \in \mathrm{Mod}_{\mathrm{F,coh}}(\mathcal{A}_X^{\mathrm{loc}})$. Then there exists a covering $(V_i)_{i \in I}$ of X by equivariant open subsets of X such that for each $i \in I$, $\mathcal{M}|_{V_i}$ is finitely generated by invariant sections.*

3.3.2 An equivalence between DQ-modules and modules over the ring of invariant sections

We now use the notion of F-action to obtain an equivalence between DQ-modules and modules over the ring of invariant sections. This equivalence will later allow us to relate the codimension-three conjectures for DQ-modules and microdifferential modules.

From now on, we assume that the action of $\mathbb{C}^\times$ is free and proper. We set $Y = X/\mathbb{C}^\times$ and write $p \colon X \to Y$ for the canonical projection.

We define the sheaf of invariant sections of $\mathcal{A}_X^{\mathrm{loc}}$ by

$$\mathcal{B}_Y(V) = \{s \in p_*\mathcal{A}_X^{\mathrm{loc}}(V) | \forall t \in \mathbb{C}^\times, \theta_t(s) = s\}.$$

This sheaf of $\mathbb{C}$-algebras is enjoying the following property which ensure that the theory of coherent sheaves over it is well-behaved as being coherent will be equivalent to be locally of finite presentation.

Theorem 3.23 ([Pet18b, Theorem 6.7])**.** *The sheaf $\mathcal{B}_Y$ is a Noetherian sheaf of $\mathbb{C}$-algebras.*

Remark 3.24. Theorem 3.23 together with [KS12, Theorem 1.3.6 (i)] ensuring that DQ-algebras are coherent imply the coherence of the sheaf of microdifferential operators on the projective cotangent bundle.

We now define the functors of globally invariant sections and equivariant extensions.

$$\begin{aligned} p_*^{\mathbb{C}^\times} : \mathrm{Mod}_{\mathrm{F}}(\mathcal{A}_X^{\mathrm{loc}}) &\to \mathrm{Mod}(\mathcal{B}_Y) \\ \mathcal{M} &\mapsto p_*^{\mathbb{C}^\times}\mathcal{M} \end{aligned}$$

where $p_*^{\mathbb{C}^\times}\mathcal{M}$ is the subsheaf of $p_*\mathcal{M}$ such that

$$p_*^{\mathbb{C}^\times}\mathcal{M}(V) := \{s \in p_*\mathcal{M}(V) \,|\, \forall t \in \mathbb{C}^\times, \phi_t(s) = s\}.$$

It follows from the definition of $\mathcal{B}_Y$, that there is a canonical morphism of sheaves of algebras

$$p^{-1}\mathcal{B}_Y \to \mathcal{A}_X^{\mathrm{loc}}.$$

This allow to define the functor

$$p^*\colon \mathrm{Mod}(\mathcal{B}_Y) \to \mathrm{Mod}(\mathcal{A}_X^{\mathrm{loc}}) \qquad \mathcal{N} \mapsto p^*\mathcal{N} = \mathcal{A}_X^{\mathrm{loc}} \otimes_{p^{-1}\mathcal{B}_Y} p^{-1}\mathcal{N}.$$

Moreover, one can shows that there is a canonical isomorphism of functors (see [Pet18b, p. 27] for details)

$$\phi\colon \mu^*p^* \simeq p_2^*p^*.$$

This allows to equip the $\mathcal{A}_X^{\mathrm{loc}}$-module $p^*\mathcal{N}$ with a canonical F-action given by

$$\phi_{\mathcal{N}}\colon \mu^*p^*\mathcal{N} \xrightarrow{\sim} p_2^*p^*\mathcal{N}.$$

Hence the following functor is well defined

$$p_{\mathbb{C}^\times}^*\colon \mathrm{Mod}(\mathcal{B}_Y) \to \mathrm{Mod}_{\mathrm{F}}(\mathcal{A}_X^{\mathrm{loc}}), \quad \mathcal{N} \mapsto p_{\mathbb{C}^\times}^*\mathcal{N} := (p^*\mathcal{N},\, \phi_{\mathcal{N}}).$$

Proposition 3.25 ([Pet18b, Proposition 6.2])**.**

$$p^*_{\mathbb{C}^\times} \colon \operatorname{Mod}(\mathcal{B}_Y) \rightleftarrows \operatorname{Mod}_{\mathrm{F}}(\mathcal{A}^{\mathrm{loc}}_X) \colon p^{\mathbb{C}^\times}_*$$

form the adjoint pair $(p^*_{\mathbb{C}^\times}, p^{\mathbb{C}^\times}_*)$.

We can now state the equivalence of categories announced in the introduction.

Theorem 3.26 ([Pet18b, Theorem 6.9])**.** *The adjoint pair* $(p^*_{\mathbb{C}^\times}, p^{\mathbb{C}^\times}_*)$ *descends to an adjunction*

$$p^*_{\mathbb{C}^\times} \colon \operatorname{Mod}_{\mathrm{coh}}(\mathcal{B}_Y) \rightleftarrows \operatorname{Mod}_{\mathrm{F,\,coh}}(\mathcal{A}^{\mathrm{loc}}_X) \colon p^{\mathbb{C}^\times}_*. \tag{3.3.3}$$

These functors are equivalences of categories inverse to each other.

3.3.3 Relation between the codimension-three conjectures

Let M be a complex manifold, T^*M its cotangent bundle and $\dot{T}^*M = T^*M \setminus M$. The canonical action of $\mathbb{C}^\times$ on $\dot{T}^*M$ is free and proper and the projective cotangent bundle is defined as $P^*M = \dot{T}^*M/\mathbb{C}^\times$. We write $p\colon \dot{T}^*M \to P^*M$ for the canonical projection. The canonical action of $\mathbb{C}^\times$ on $\dot{T}^*M$ lift to a weight one F-action on $\widehat{\mathcal{W}}_{\dot{T}^*M}$.

Let X be a conical open subset of $\dot{T}^*M$ and set $Y = p(X)$. Then the sheaf of globally invariant sections of $\widehat{\mathcal{W}}_X$, $p^{\mathbb{C}^\times}_*\widehat{\mathcal{W}}_X$ is isomorphic to $\widehat{\mathcal{E}}_{P^*M}|_Y$ that we shorten in $\widehat{\mathcal{E}}_Y$. Then, applying Theorem 3.26 to the above setting, we obtain the following equivalence of categories (details can be found in [Pet18b, sub-section 6.4]).

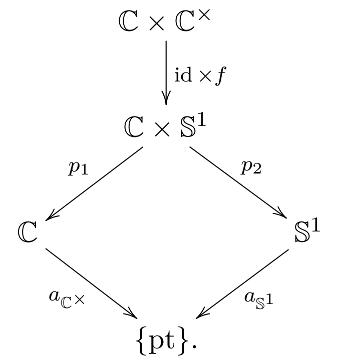
Proposition 3.27 ([Pet18b, Proposition 6.14])**.** *Assume that X does not intersect the zero sections of T^*M and let $Y = X/\mathbb{C}^\times$. Then the adjoint pair $(p^*_{\mathbb{C}^\times}, p^{\mathbb{C}^\times}_*)$ induces a well-defined adjunction*

$$p^*_{\mathbb{C}^\times} \colon \operatorname{Mod}_{\mathrm{coh}}(\widehat{\mathcal{E}}_Y) \rightleftarrows \operatorname{Mod}_{\mathrm{F,\,coh}}(\widehat{\mathcal{W}}_X) \colon p^{\mathbb{C}^\times}_*.$$

These functors are equivalences of categories inverse to each other.

Remark 3.28. It is not possible in general to obtain an equivalence between the categories $\operatorname{Mod}_{\mathrm{F,\,coh}}(\widehat{\mathcal{W}}_X)$ and $\operatorname{Mod}_{\mathrm{coh}}(\widehat{\mathcal{E}}_X)$ since $\mathbb{C}^\times$ is not simply connected. We illustrate this in the following example

Example 3.29. Consider $M = \mathbb{C}$ endowed with the coordinate (x) and set $X = \dot{T}^*M$. Let $\Lambda = \dot{T}^*_0M \simeq \mathbb{C}^\times$ and $f\colon \mathbb{C}^\times \to \mathbb{S}^1$, $z \mapsto z/|z|$. We have the following commutative diagram



We also set $p = p_1 \circ id \times f$ and $q = p_2 \circ id \times f$. Consider the $\widehat{\mathcal{E}}_{P^*M}$-module $\mathcal{C}_{\{0\}|P^*M} = \widehat{\mathcal{E}}_{P^*M}/\widehat{\mathcal{E}}_{P^*M}x$. The module $\mathcal{C}_{\{0\}|X} = p^{-1}\mathcal{C}_{\{0\}|P^*M}$ is supported on Λ. Let L be a non-trivial rank one local system on $\mathbb{S}^1$. Consider the coherent $\widehat{\mathcal{E}}_X$-module

$$\mathcal{C}_{\{0\}|X} \otimes q^{-1}L.$$

Assume that there exists a coherent DQ-module $\mathcal{M}$ on X endowed with an F-action such that its sheaf of invariant sections $\mathcal{M}^{\mathbb{C}^\times} \simeq \mathcal{C}_{\{0\}|X} \otimes q^{-1}L$. By Proposition 3.22, $\mathcal{M}$ would have a globally invariant section. This leads to a contradiction since $\Gamma(X; \mathcal{C}_{\{0\}|X} \otimes q^{-1}L) \simeq 0$. Indeed,

$$\begin{aligned} p_*(\mathcal{C}_{\{0\}|X} \otimes q^{-1}L) &\simeq p_{1*}(\mathrm{id} \times f)_*((\mathrm{id} \times f)^{-1}p_1^{-1}\mathcal{C}_{\{0\}|P^*M} \otimes (\mathrm{id} \times f)^{-1}p_2^{-1}L) \\ &\simeq p_{1*}(\mathrm{id} \times f)_*(\mathrm{id} \times f)^{-1}(p_1^{-1}\mathcal{C}_{\{0\}|P^*M} \otimes p_2^{-1}L) \\ &\simeq p_{1*}(p_1^{-1}\mathcal{C}_{\{0\}|P^*M} \otimes p_2^{-1}L) \\ &\simeq p_{1*}(p_1^{-1}\mathcal{C}_{\{0\}|P^*M} \otimes p_2^{-1}L) \\ &\simeq \mathcal{C}_{\{0\}|P^*M} \otimes p_{1*}p_2^{-1}L \simeq \mathcal{C}_{\{0\}|P^*M} \otimes a_{\mathbb{C}^\times}^{-1} a_{\mathbb{S}^1 *}L. \end{aligned}$$

and $\Gamma(X; L) \simeq 0$.

We now go back to the codimension-three conjecture and briefly sketch, how to recover the codimension-three conjecture for microdifferential modules from the one for DQ-modules. For that purpose, we need to be able to extend F-actions through analytic subset. The next lemma, whose proof relies on our functorial formulation of an F-action, ensures that this is possible under the hypothesis of the codimension-three conjecture for holonomic DQ-modules.

Lemma 3.30 ([Pet18b, Lemma 7.4])**.** *Assume that X is conical and does not intersect the zero section. Let Λ be a conical Lagrangian subvariety of X, let Z be a closed conical analytic subset of Λ such that $\operatorname{codim}_\Lambda Z \geq 2$, $j \colon X \setminus Z \hookrightarrow X$ be the inclusion and $\mathcal{M}$ a holonomic $\widehat{\mathcal{W}}_X$-module supported in Λ such that $\mathcal{M} \in \mathrm{Mod}_{\mathrm{F}}(\widehat{\mathcal{W}}_X|_{X \setminus Z})$. Then $\mathcal{M} \in \mathrm{Mod}_{\mathrm{F}}(\widehat{\mathcal{W}}_X)$.*

As observed in [KV14], it is sufficient, by the dummy variable trick, to prove the codimension-three conjecture for holonomic microdifferential modules on an open subset of Y of P^*X. Using Proposition 3.27, we reduce the question to the case of holonomic DQ-modules endowed with an F-action. The codimension-three conjecture for holonomic DQ-modules is already proved. We still need to extend the F-action to make to connection with the microdifferential case. But Lemma 3.30 ensure that an holonomic DQ-modules endowed with a holomorphic F-action and satisfying the hypothesis of the codimension-three conjecture extends uniquely to X into an holonomic DQ-modules endowed with an F-action. Using Proposition 3.27 one more time, we get the result for microdifferential modules.

Chapter 4

Distances for persistence modules

4.1 Introduction

In this chapter, we report on the work [BP18] and [PS20]. Both of these papers were motivated by considerations arising from topological data analysis. These articles deal with distances on categories of sheaves. Persistent homology allows to summarize, in a scale independent way, the topological features of points clouds. One of the key element for the applications of persistent homology is that the space of persistent modules can be endowed with various distances which allow to quantify the dissimilarities of the data. These distances were defined for Alexandrov sheaves on a vector space endowed with a linear order (or equivalently persistence modules) [CCSG$^+$09, Cur14, Les15, dSMS18], while in [KS18a], the authors introduced a new distance on the derived categories of sheaves on a finite dimensional vector space (endowed with the euclidean topology) often called the convolution distance. Simultaneously, several categorical framework were proposed to study interleaving distances [BdSS14, dSMS18, Sco20] but were not providing indications on how to construct an interleaving distance for sheaves on a metric spaces. This is what we achieved, in [PS20], where we constructed an interleaving distance on the bounded derived category of abelian sheaves on a large class of metric spaces. This class contains in particular the metric space associated with Riemann manifolds of strictly positive convexity radius. With this distance at hand, we study the metric properties of integral transforms. We study Lipschitz kernel and established that the kernel associated with a Lipschitz map is a Lipschitz kernel. With Nicolas Berkouk, we formulated a conjecture reducing the computation of the convolution distance of multi-persitence modules to the computation of a supremum involving only the convolution distance for sheaves on $\mathbb{R}$. This conjecture together with the structure theorem for constructible sheaves [Gui19] and the isometry theorem for the convolution distance identifying it with a matching distance [BG18] suggests a way to numerically approximate the convolution distance for multipersistence module. In [PS20], We also prove that the Fourier-Sato transform and the Radon transform are isometries.

As already hinted, there are several framework to approach the study of persistence modules and their metric properties. It is natural to compare them. From a sheaf theoretic point of view the problem boils down to compare Alexandrov sheaves endowed with the interleaving with γ-sheaves with the interleaving distances and γ-sheaves equipped with the convolution distance. In [BP18], we proved, generalizing results of [CCBdS16b], that the category of γ-sheaves is a reflexive localization of Alexandrov sheaves and that this localization is isometric. We also established that under some conditions, always satisfied by γ-sheaves originating from finite clouds of points, the interleaving and convolution distances are equals. In this text, we remove these conditions by relying on a result of [PS20].

4.2 Distances for categories of sheaves

4.2.1 Integral transform and convolution

In this section, we set up a few notation and present an associativity criterion for non-proper composition of kernels.

Given topological spaces X_i ($i = 1,\ 2,\ 3$), we write X_{ij} for $X_i \times X_j$, X_{123} for $X_1 \times X_2 \times X_3$, $p_i \colon X_{ij} \to X_i$ and $p_{ij} \colon X_{123} \to X_{ij}$ for the projections. One defines the composition of kernels for $K_{ij} \in \mathrm{D^b}(\mathbf{k}_{X_{ij}})$ as

$$K_{12} \underset{2}{\circ} K_{23} := \mathrm{R}p_{13!}(p_{12}^{-1} K_{12} \otimes p_{23}^{-1} K_{23}),$$
$$K_{12} \overset{\mathrm{np}}{\underset{2}{\circ}} K_{23} := \mathrm{R}p_{13*}(p_{12}^{-1} K_{12} \otimes p_{23}^{-1} K_{23}).$$

The proper composition of kernel $- \underset{2}{\circ} -$ is associative. This is not the case of the non-proper one $- \overset{\mathrm{np}}{\underset{2}{\circ}} -$. Nonetheless, we have the following result.

Theorem 4.1 ([PS20, Theorem 2.1.8])**.** *Let X_i ($i = 1, 2, 3, 4$), be four C^∞-manifolds and let $K_i \in \mathrm{D^b}(\mathbf{k}_{X_{i,i+1}})$, ($i = 1, 2, 3$). Assume that K_1 is cohomologically constructible, q_1 is proper on* $\mathrm{supp}(K_1)$ *and* $\mathrm{SS}(K_1) \cap (T^*_{X_1} X_1 \times T^* X_2) \subset T^*_{X_{12}} X_{12}$. *Then*

$$K_1 \overset{\mathrm{np}}{\underset{2}{\circ}} (K_2 \overset{\mathrm{np}}{\underset{3}{\circ}} K_3) \simeq (K_1 \overset{\mathrm{np}}{\underset{2}{\circ}} K_2) \overset{\mathrm{np}}{\underset{3}{\circ}} K_3.$$

When the topological space X is a finite dimensional vector space, one defines the convolution of sheaves. We write $s \colon \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ for the addition map. If $F,\ G \in \mathrm{D^b}(\mathbf{k}_{\mathbb{V}})$, we set

$$F \star G := \mathrm{R}s_!(F \boxtimes G),$$
$$F \overset{\mathrm{np}}{\star} F := \mathrm{R}s_*(F \boxtimes G).$$

We consider the morphism

$$u : \mathbb{V} \times \mathbb{V} \to \mathbb{V}, \quad (x, y) \mapsto x - y.$$

We will need the following elementary formula relating non-proper convolution and non-proper composition.

Lemma 4.2. *Let F, $G \in \mathrm{D^b}(\mathbf{k}_{\mathbb{V}})$. Then*

(i) $(u^{-1}F) \overset{\mathrm{np}}{\circ} G \simeq F \overset{\mathrm{np}}{\star} G$

(ii) $u^{-1}F \overset{\mathrm{np}}{\circ} u^{-1}G \simeq u^{-1}(F \overset{\mathrm{np}}{\star} G)$

Proof. We will only prove (i), the proof of (ii) being similar.

$$\begin{aligned} u^{-1}F \overset{\mathrm{np}}{\circ} G &\simeq \mathrm{R}p_{1*}(u^{-1}F \otimes p_2^{-1}G) \\ &\simeq \mathrm{R}p_{1*}((u \times \mathrm{id})^{-1}p_1^{-1}F \otimes (u \times \mathrm{id})^{-1}p_2^{-1}G) \\ &\simeq \mathrm{R}p_{1*}(u \times \mathrm{id})^{-1}(F \boxtimes G) \\ &\simeq \mathrm{R}p_{1*}\mathrm{R}(s \times \mathrm{id})_*(F \boxtimes G) \\ &\simeq F \overset{\mathrm{np}}{\star} G. \end{aligned}$$

□

4.2.2 Thickening of the diagonal

In this section, we explain how to construct an interleaving distance called the convolution distance on the bounded derived categories of abelian sheaves on a good metric spaces (X, d_X).

Recall that a topological space is good if it is Hausdorff, locally compact, countable at infinity and of finite flabby dimension. Let (X, d_X) be a metric space. For $a \geq 0$, $x_0 \in X$, set

$$\begin{aligned} B_a(x_0) = B(x_0, a) &= \{x \in X; d_X(x_0, x) \leq a\}, \\ \Delta_a &= \{(x_1, x_2) \in X \times X; d_X(x_1, x_2) \leq a\}, \\ \Gamma_d &= \{(x_1, x_2, t) \in X \times X \times \mathbb{R}_{\geq 0}; d_X(x_1, x_2) \leq t\}. \end{aligned}$$

If Z is a locally closed subset of X, we denote by $\mathbf{k}_Z$ the sheaf associated with Z

Definition 4.3. A metric space (X, d_X) is good if the underlying topological space is good and moreover there exists some $\alpha_X > 0$ such that for all $0 \leq a, b$ with $a + b \leq \alpha_X$, one has

$$\begin{cases}\text{(i) for any } x \in X,\ B(x,a) \text{ is contractible,}\\ \text{(ii) the two projections } q_1 \text{ and } q_2 \text{ are proper on } \Delta_a,\\ \text{(iii) } \Delta_a \circ \Delta_b = \Delta_{a+b},\\ \text{(iv) for any } x_1, x_2 \in X,\ B(x_1,a)\cap B(x_2,b) \text{ is contractible or empty.}\end{cases} \tag{4.2.1}$$

Clearly, in this definition, α_X is not unique. If we want to precise a choice of α_X for a good metric space, we write (X, d_X, α_X).

We consider the family of kernels $(\mathbf{k}_{\Delta_a})_{0\leq a\leq \alpha_X}$. The goodness hypothesis on the metric space ensure that $\mathbf{k}_{\Delta_a}\circ\mathbf{k}_{\Delta_b}\simeq\mathbf{k}_{\Delta_{a+b}}$ for $a+b\leq\alpha_X$.

The convolution distance is defined by mean of a kernel $\mathfrak{K}_a$ such for a sufficiently small, $\mathfrak{K}_a = \mathbf{k}_{\Delta_a}$ and $\mathfrak{K}_a$ is defined by composing iteratively for larger value of a. That is for a good metric space (X, d_X, α_X), we set $\lambda = \frac{\alpha_X}{2}$ and for $a \geq 0$, we write $a = n\lambda + r_a$ with $0 \leq r_a < \lambda$ and sets

$$\mathfrak{K}_a := \underbrace{\mathbf{k}_{\Delta_\lambda}\circ\cdots\ \circ\mathbf{k}_{\Delta_\lambda}}_{n}\circ\ \mathbf{k}_{\Delta_{r_a}}.$$

Following a similar strategy, we define the restriction morphisms $\rho_{a,b}\colon \mathfrak{K}_b \to \mathfrak{K}_a$ for $a \leq b$ satisfying the following compatibility conditions $\rho_{a,b}\circ\rho_{b,c} = \rho_{a,c}$ for $a \leq b \leq c$ and $\rho_{a,a} = \mathrm{id}$.

This construction is formalized through the notion of monoidal presheaves (see [PS20, Definition 2.1.1]) and Theorem 2.2.2 of [PS20]). In particular, regarding the ordered set $(\mathbb{R}_{\geq 0}, \leq, +)$ as a monoidal category with unit that we simply denote by $\mathbb{R}_{\geq 0}$ we obtain (see [PS20, §2.3]) a monoidal functor

$$\mathfrak{K}\colon \mathbb{R}_{\geq 0} \to (\mathrm{D^b}(\mathbf{k}_{X\times X}), \circ), \quad a \mapsto \mathfrak{K}_a$$

such that for $a \leq \alpha_X$, $\mathfrak{K}_a \simeq \mathbf{k}_{\Delta_a}$. This last properties determine the functor $\mathfrak{K}$ up to isomorphism.

These results can be summarized in the following result

Theorem 4.4 ([PS20, Theorem 2.3.4])**.** *There exists an object* $\mathfrak{K}_{\mathrm{dist}} \in \mathrm{D^{lb}}(\mathbf{k}_{X\times X\times\mathbb{R}_{\geq 0}})$ *such that*

(i) $\mathfrak{K}_{\mathrm{dist}}|_{\{t=a\}} \simeq \mathfrak{K}_a$*, for all* $a \geq 0$*,*

(ii) $\mathfrak{K}_{\mathrm{dist}}|_{[0,\alpha_X]} \simeq \mathbf{k}_{\Gamma_d}|_{[0,\alpha_X]}$.

Moreover, such an object satisfying (i)–(ii) *is unique up to isomorphism.*

4.2.3 The convolution distance

With Pierre Schapira, we generalized the convolution distance introduce in [KS18a] for sheaves on $\mathbb{V}$ to good metric spaces. This distance is a kind of sheafy version of the

Hausdorff distance.

We define the functor

$$\mathfrak{L}_a = \Phi_{\mathfrak{K}_a} = \mathrm{R}q_{1!}(\mathfrak{K}_a \otimes q_2^{-1}(\,\bullet\,)).$$

This functor allow us to define the convolution distance.

Definition 4.5. Let $F, G \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ and let $a \geq 0$.

(a) One says that F and G are *a-isomorphic* if there are morphisms $f\colon \mathfrak{L}_a(F) \to G$ and $g\colon \mathfrak{L}_a(G) \to F$ which satisfies the following compatibility conditions: the composition $\mathfrak{L}_{2a}(F) \xrightarrow{\mathfrak{L}_a f} \mathfrak{L}_a(G) \xrightarrow{g} F$ and the composition $\mathfrak{L}_{2a}(G) \xrightarrow{\mathfrak{L}_a g} \mathfrak{L}_a(F) \xrightarrow{f} G$ coincide with the morphisms induced by the canonical morphism $\rho_{0,2a}\colon \mathfrak{K}_{2a} \to \mathfrak{K}_0$.

(b) One sets $\mathrm{dist}_X(F, G) = \inf\Big(\{+\infty\} \cup \{a \in \mathbb{R}_{\geq 0}\,;\, F \text{ and } G \text{ are } a\text{-isomorphic}\}\Big)$ and calls $\mathrm{dist}_X(\,\bullet\,,\,\bullet\,)$ the convolution distance.

Note that for $F, G, H \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$,

- F and G are 0-isomorphic if and only if $F \simeq G$,
- $\mathrm{dist}_X(F, G) = \mathrm{dist}_X(G, F)$,
- $\mathrm{dist}_X(F, G) \leq \mathrm{dist}_X(F, H) + \mathrm{dist}_X(H, G)$.

Remark 4.6. As already mentioned, the convolution distance was initially introduced for sheaves on a finite dimensional real vector space $\mathbb{V}$ endowed with a norm $\|\cdot\|$. In this setting, the convolution distance can be express in term of convolution by the sheaves associated to closed ball centered in 0. That is, writing B_a instead of $B_a(0) = \{x \in \mathbb{V}; \|x\| \leq a\}$, we have

$$\mathfrak{L}_a \simeq \mathbf{k}_{B_a} \star (\cdot).$$

4.2.4 The case of Riemannian manifolds

Consider a Riemannian manifold (X, g) of class C^∞ and denote by d_X its associated distance. We denote by r_{conv} the convexity radius of (X, g).

$$\begin{cases} \text{We shall assume that } (X,g) \text{ is complete and has a strictly positive} \\ \text{convexity radius } r_{\mathrm{conv}}. \text{ Then we choose } 0 < \alpha_X < r_{\mathrm{conv}}. \end{cases} \tag{4.2.2}$$

Note that compact Riemannian manifolds satisfy (4.2.2).

Theorem 4.7 ([PS20, Theorem 3.1.2])**.** *Let (X, g) be a Riemannian manifold satisfying* (4.2.2)*. Then*

(a) Hypothesis (4.2.1) *is satisfied i.e. (X, d_X) is a good metric space.*

(b) (i) *For* $0 < a \leq \alpha_X$, *the open set* Δ_a° *is locally cohomologically trivial, the sheaves* $\mathbf{k}_{\Delta_a}$ *and* $\mathbf{k}_{\Delta_a^\circ}$ *are cohomologically constructible and*

$$\mathrm{D}'_{X\times X}\mathbf{k}_{\Delta_a^\circ} \simeq \mathbf{k}_{\Delta_a}, \quad \mathrm{D}'_{X\times X}\mathbf{k}_{\Delta_a} \simeq \mathbf{k}_{\Delta_a^\circ}.$$

(ii) *For* $a \geq 0$, $\mathrm{SS}(\mathfrak{K}_a) \cap (T^*_X X \times T^*X \cup T^*X \times T^*_X X) \subset T^*_{X\times X}(X \times X)$.

Part (b) of Theorem 4.7 allows to apply Theorem 4.1 which ensure the associativity of non-proper convolution.

Finally, for Riemannian Manifolds we obtain that

Proposition 4.8 ([PS20, Corollary 3.1.3])**.** *Let* (X, g) *be a Riemannian manifold satisfying* (4.2.2)*. Then for* $b \in \mathbb{R}$, *the functor* $\mathfrak{L}_b\colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ *is an equivalenvce of categories and an isometry for the pseudo-distance* dist_X.

4.2.5 Integral transforms

In this subsection, we present some of the properties of the Fourier-Sato transform with respects to the convolution distance. We also study Lipschitz kernel.

Fourier Sato transform

Using the canonical Riemannian structure on the sphere and the dual sphere, we provided, in [PS20], a new formulation of the kernel of the Fourier-Sato transform and proved that this integral transform is an isometry.

Consider the topological n-sphere ($n > 0$). That is the topological space obtained as follows. Let $\mathbb{V}$ be a real vector space of dimension $n+1$, set $\dot{\mathbb{V}} = \mathbb{V} \setminus \{0\}$ and $\mathbf{S} := \dot{\mathbb{V}}/\mathbb{R}^+$ where $\mathbb{R}^+$ is the multiplicative group $\mathbb{R}_{>0}$. The dual sphere $\mathbf{S}^*$ is defined similarly using $\mathbb{V}^*$ instead of $\mathbb{V}$. The sets

$$P = \{(y,x) \in \mathbf{S}^* \times \mathbf{S}; \langle y, x\rangle \geq 0\}, \quad I = \{(y,x) \in \mathbf{S}^* \times \mathbf{S}; \langle y, x\rangle > 0\}, \tag{4.2.3}$$

are well-defined. We define the kernel

$$K_I = \mathbf{k}_I \otimes (\omega_{\mathbf{S}^*} \boxtimes \mathbf{k}_{\mathbf{S}}).$$

The Fourier-Sato transform $\mathfrak{F}^\wedge$ and its inverse $\mathfrak{F}^\vee$ are the functors

$$\mathfrak{F}^\wedge := \mathbf{k}_P \circ \colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbf{S}}) \rightleftarrows \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbf{S}^*}) \colon \circ K_I =: \mathfrak{F}^\vee.$$

We have the following classical result

Theorem 4.9 ([SKK73])**.** *The functor* $\mathfrak{F}^\wedge$ *and the functor* $\mathfrak{F}^\vee$ *are equivalences of categories quasi-inverse to each other.*

Now, we consider the n-sphere $\mathbb{S}^n$ of radius 1 embedded in the Euclidian space $\mathbb{R}^{n+1}$ and endowed with its canonical Riemannian metric. The topological sphere $\mathbf{S}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^+$ and the Euclidian sphere $\mathbb{S}^n$ are identified via the map

$$\mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{S}^n, \quad x \mapsto x/||x||$$

where $||\cdot||$ denotes the Euclidian norm on $\mathbb{R}^{n+1}$. Moreover, the isomorphism $\mathbb{R}^n \to \mathbb{R}^{n*}$, mapping the canonical basis to its dual basis induces an isomorphism $\mathbb{S}^n \simeq \mathbb{S}^{n*}$ and we use it to identify these spaces.

Theorem 4.10 ([PS20, Theorem 5.1.4]). *The equivalence* $\mathfrak{F}^\wedge$ *given by* Theorem 4.9 *induces an isometry*

$$\mathfrak{F}^\wedge \colon (\mathrm{D^b}(\mathbf{k}_{\mathbb{S}}), \mathrm{dist}_{\mathbb{S}}) \xrightarrow{\sim} (\mathrm{D^b}(\mathbf{k}_{\mathbb{S}^*}), \mathrm{dist}_{\mathbb{S}^*}).$$

Proof. Let us identify $\mathbb{S}^n$ and the dual sphere $\mathbb{S}^{n*}$. Then the set P of (4.2.3) may be also defined as:

$$P = \{(x,y) \in \mathbb{S} \times \mathbb{S}; d_{\mathbb{S}}(x,y) \leq \pi/2\}$$

Since $\mathbf{k}_{\Delta_{\pi/2}} \simeq \mathbf{k}_{\Delta_{\pi/4}} \circ \mathbf{k}_{\Delta_{\pi/4}}$ we have $\mathbf{k}_P \simeq \mathfrak{K}_{\pi/2}$. (It was not possible to deduce directly this result form (4.2.5) since $\alpha_{\mathbb{S}} < r_{\mathrm{conv}}(\mathbb{S}) = \pi/2$.) Therefore $\mathbf{k}_P \circ$ is an isometry by Proposition 4.8. ☐

In [PS20], we also obtained similar results for the Radon transform.

Lipschitz kernel

Definition 4.11. Let $\delta > 0$ and let $K \in \mathrm{D^b}(\mathbf{k}_{Y\times X})$. We say that K is a δ-Lipschitz kernel if there exists $\rho > 0$ such that $\rho \leq \alpha_X$ and $\delta\rho \leq \alpha_Y$ and there are morphisms of sheaves $\sigma_a \colon \mathfrak{K}^Y_{\delta a} \circ K \to K \circ \mathfrak{K}^X_a$ for $0 \leq a \leq \rho$ satisfying the following compatibility relations:

(i) for $0 \leq a \leq b \leq \rho$, the diagram of sheaves commutes:

$$\begin{array}{ccc} \mathfrak{K}^Y_{\delta b} \circ K & \xrightarrow{\sigma_b} & K \circ \mathfrak{K}^X_b \\ \downarrow{\scriptstyle \rho^Y_{\delta a,\delta b}} & & \downarrow{\scriptstyle \rho^X_{a,b}} \\ \mathfrak{K}^Y_{\delta a} \circ K & \xrightarrow{\sigma_a} & K \circ \mathfrak{K}^X_a, \end{array}$$

(ii) for $0 \leq a$, b and $a+b \leq \rho$, the diagram of sheaves commutes:

$$\mathfrak{K}^Y_{\delta(a+b)} \circ K \xrightarrow{\mathfrak{K}^Y_{\delta b} \circ \sigma_a} \mathfrak{K}^Y_{\delta b} \circ K \circ \mathfrak{K}^X_a \xrightarrow{\sigma_b \circ \mathfrak{K}^X_a} K \circ \mathfrak{K}^X_{a+b}, \qquad \sigma_{a+b} \colon \mathfrak{K}^Y_{\delta(a+b)} \circ K \to K \circ \mathfrak{K}^X_{a+b}.$$

A Lipschitz kernel is a δ-Lipschitz kernel for some $\delta > 0$.

The next result is [PS20, Theorem 4.2.4] and is a restatement of [dSMS18, Theorem 4.3] in the language of kernels.

Theorem 4.12 (The functorial Lipschitz theorem)**.** *Let (X, d_X) and (Y, d_Y) be good metric spaces and let $K \in \mathrm{D^b}(\mathbf{k}_{Y\times X})$ be a δ-Lipschitz kernel. Let $F_1, F_2 \in \mathrm{D^b}(\mathbf{k}_X)$.*

(i) One has $\mathrm{dist}_Y(K \circ F_1, K \circ F_2) \leq \delta \cdot \mathrm{dist}_X(F_1, F_2)$.

(ii) Assume moreover that both (X, d_X) and (Y, d_Y) are Riemannian manifolds satisfying (4.2.2). Then $\mathrm{dist}_Y(K \overset{\mathrm{np}}{\circ} F_1, K \overset{\mathrm{np}}{\circ} F_2) \leq \delta \cdot \mathrm{dist}_X(F_1, F_2)$.

In [PS20], we proved the following.

Proposition 4.13 ([PS20, Corollary 4.2.7])**.** *Let $f\colon (X, d_X) \to (Y, d_Y)$ be a δ-Lipschitz map. Then the kernel $\mathbf{k}_{\Gamma_f}$ is δ-Lipschitz.*

The above result implies that

Corollary 4.14 ([PS20, Corollary 4.2.9])**.** *Let (X, d_X) and (Y, d_Y) be good metric spaces and let $f\colon X \to Y$ be a δ-Lipschitz map. Let $F_1, F_2 \in \mathrm{D^b}(\mathbf{k}_X)$,*

(i) One has $\mathrm{dist}_Y(\mathrm{R}f_! F_1, \mathrm{R}f_! F_2) \leq \delta \cdot \mathrm{dist}_X(F_1, F_2)$.

(ii) Assume moreover that both (X, d_X) and (Y, d_Y) are Riemannian manifolds satisfying (4.2.2). Then $\mathrm{dist}_Y(\mathrm{R}f_* F_1, \mathrm{R}f_* F_2) \leq \delta \cdot \mathrm{dist}_X(F_1, F_2)$.

A conjecture

Let $\phi\colon \mathbb{V} \to \mathbb{R}$ be a Lipschitz function. The Lipschitz modulus of ϕ is the real number $L(\phi) = \sup_{x \neq y} \left(\frac{|\phi(x) - \phi(y)|}{\|x - y\|}\right)$. In particular for every $x,\ y \in \mathbb{V}$

$$|\phi(x) - \phi(y)| \leq L(\phi)\|x - y\|.$$

We denote by $\mathrm{Lip}_1(\mathbb{V})$ (resp. $\mathrm{Lip}_{\leq 1}(\mathbb{V})$), the set of Lipschitz functions on $\mathbb{V}$ of Lipschitz modulus equal to one (resp. less or equal to one). Corollary 4.14 implies that for $\phi \in \mathrm{Lip}_1(\mathbb{V})$,

$$\mathrm{dist}_{\mathbb{R}}(\phi_! F, \phi_! G) \leq \mathrm{dist}_{\mathbb{V}}(F, G).$$

This leads Nicolas Berkouk and myself to the following conjecture.

Conjecture 4.15 (Berkouk-P.)**.** Let $F, G \in \mathrm{D^b}(\mathbf{k}_{\mathbb{V}})$. Then

$$\mathrm{dist}_{\mathbb{V}}(F, G) = \sup_{\phi \in \mathrm{Lip}_1} (\mathrm{dist}_{\mathbb{R}}(\phi_! F, \phi_! G))$$

More generally, it is quite natural to study the following distances.

$$d_{\mathfrak{F}}(F,G) = \sup_{\phi\in\mathfrak{F}}(\mathrm{dist}_{\mathbb{R}}(\phi_! F, \phi_! G))$$

where $\mathfrak{F}$ is a subset of $\mathrm{Lip}_{\leq 1}(\mathbb{V})$ (or more generally a subset of a space of functions). It is interesting to remark that the distances of the form $d_{\mathfrak{F}}$ are stable by constructions. The distance $d_{\mathfrak{F}}$ for $\mathfrak{F} = \mathrm{Lip}_1(\mathbb{V}) \cap \mathbb{V}^*$ seems to be of particular interests and we are currently studying it. Note that it is also possible to consider the above construction with $\phi_!$ replace by ϕ_*. One also checks that

$$\sup_{\phi\in\mathrm{Lip}_1}(\mathrm{dist}_{\mathbb{R}}(\phi_! F, \phi_! G)) = \sup_{\phi\in\mathrm{Lip}_{\leq 1}}(\mathrm{dist}_{\mathbb{R}}(\phi_! F, \phi_! G)).$$

This conjecture together with the isometry theorem of [BG18] showing that in dimension one the convolution distance is a matching distance may help to construct new distances (in addition to the weighted bottleneck distances) for multipersistence modules easier to compute than the convolution distance.

4.3 Comparison of distances

There exists several possible formulations of the notion of persistence modules together with their respective distances.

They can be described as functor over posets, n-graded modules. These two formulations are due to Gunnar Carlson and Afra Zomorodian and are known to be equivalent and isometric [CZ09]. We know, after the work of Justin Curry (see [KS18a] for a proof), that these two formulations are isometrically equivalent to the one where the category of persistence modules corresponds to the category of Alexandrov sheaves endowed with the interleaving distances. With Nicolas Berkouk, we provide in [BP18] a comparison of the approach of persistence via Alexandrov sheaves to the approach in terms of γ-sheaves introduced by Masaki Kashiwara and Pierre Schapira in [KS18a]. In particular, we show that these different approaches are isometric and that γ-sheaves can be obtain as the quotient of Alexandrov sheaves by the so-called ephemeral modules. This generalized a result of [CCBdS16a] to the multipersistent setting.

4.3.1 γ and Alexandrov sheaves

γ-sheaves

Let $\mathbb{V}$ be a real finite dimensional vector space and let γ be a closed proper convex cone with non-empty interior.

If A is a subset of $\mathbb{V}$, we write $\mathrm{Int}(A)$ for the interior of A in the usual topology of $\mathbb{V}$

and A° for the polar of A, that is the set

$$A^\circ = \{\xi \in \mathbb{V}^* \,|\, \text{for all } v \in A, \langle \xi, v\rangle \geq 0\}.$$

We say that a subset A of $\mathbb{V}$ is *γ-invariant* if $A = A + \gamma$. The set of γ-invariant open subsets of $\mathbb{V}$ (endowed with the euclidean topology) form a topology on $\mathbb{V}$ called the γ-topology. We denote by $\mathbb{V}_\gamma$ the vector space $\mathbb{V}$ endowed with the γ-topology. We write $\phi_\gamma \colon \mathbb{V} \to \mathbb{V}_\gamma$ for the continuous map whose underlying function is the identity. The family $\{x + \mathrm{Int}(\gamma)\}_{x \in \mathbb{V}_\gamma}$ is a basis for the γ-topology. We denote by $(\cdot)^a \colon \mathbb{V} \to \mathbb{V}$, $x \mapsto -x$ the antipodal map.

Following [KS18a], we set

$$\begin{aligned} \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ,a}}(\mathbf{k}_\mathbb{V}) &= \{F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_\mathbb{V}) \,|\, \mathrm{SS}(F) \subset \gamma^{\circ,a}\},\\ \mathrm{Mod}_{\gamma^{\circ,a}}(\mathbf{k}_\mathbb{V}) &= \mathrm{Mod}(\mathbf{k}_\mathbb{V}) \cap \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ,a}}(\mathbf{k}_\mathbb{V}). \end{aligned}$$

and recall the following result.

Theorem 4.16 ([KS18a, Theorem 1.5])**.** *Let γ be a proper closed convex cone in $\mathbb{V}$. The functor $\mathrm{R}\phi_{\gamma *} \colon \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ,a}}(\mathbf{k}_\mathbb{V}) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}_\gamma})$ is an equivalence of triangulated categories with quasi-inverse ϕ_γ^{-1}.*

This result implies that the bounded category of γ-sheaves can be identified with a full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_\mathbb{V})$.

There is the following microlocal characterization of γ-sheaves via the microlocal cut-off lemma. The canonical map $\mathbf{k}_{\gamma^a} \to \mathbf{k}_{\{0\}}$ induces a morphism

$$F \overset{\mathrm{np}}{\star} \mathbf{k}_{\gamma^a} \to F. \tag{4.3.1}$$

Proposition 4.17 ([GS14, Proposition 3.9])**.** *Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_\mathbb{V})$. Then $F \in \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ,a}}(\mathbf{k}_\mathbb{V})$ if and only if the morphism* (4.3.1) *is an isomorphism.*

Alexandrov sheaves

Let γ be a closed proper convex cone in $\mathbb{V}$. The datum of γ endows $\mathbb{V}$ with the order

$$x \leq_\gamma y \text{ if and only if } x + \gamma \subset y + \gamma.$$

A lower (resp. upper) set U of $(\mathbb{V}, \leq_\gamma)$ is a subset of $\mathbb{V}$ such that if $x \in U$ and $y \in \mathbb{V}$ with $y \leq_\gamma x$ (resp. $x \leq_\gamma y$) then $y \in U$. By convention, the Alexandrov topology on $(\mathbb{V}, \leq_\gamma)$ is the topology whose open sets are the lower sets. A basis of this topology is given by the sets of the form $x + \gamma$ for $x \in \mathbb{V}$. We denote by $\mathbb{V}_\mathfrak{a}$, the vector space $\mathbb{V}$ endowed with the Alexandrov topology induced by the order $\leq_\gamma$.

Comparison between Alexandrov and γ-sheaves

We introduce the following morphism of sites. First, the morphism $\alpha\colon \mathbb{V}_\gamma \to \mathbb{V}_\mathfrak{a}$ defined by

$$\alpha^t\colon \operatorname{Op}(\mathbb{V}_\mathfrak{a}) \to \operatorname{Op}(\mathbb{V}_\gamma), \quad U = \bigcup_{x \in U} x + \gamma \mapsto \bigcup_{x \in U} x + \operatorname{Int}(\gamma).$$

and

$$\beta\colon \mathbb{V}_\mathfrak{a} \to \mathbb{V}_\gamma, \quad \beta^t(x + \operatorname{Int}(\gamma)) = x + \operatorname{Int}(\gamma).$$

The morphism of sites α and β provide the following adjunctions

$$\alpha^{-1}\colon \mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}}) \rightleftarrows \mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})\colon \alpha_*,$$
$$\beta^{-1}\colon \mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma}) \rightleftarrows \mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})\colon \beta_*.$$

Proposition 4.18 ([BP18, Proposition 2.11 & 3.10])**.** *(i) There is a canonical isomorphisms of functors $\alpha^{-1} \simeq \beta_*$,*

(ii) the functor $\mathrm{R}\alpha_*$ is fully faithful,

(iii) the functor β^{-1} is fully faithful.

We now consider the full subcategory of $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$

$$\operatorname{Ker}\alpha^{-1} = \{F \in \mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}}) \mid \alpha^{-1}F \simeq 0\}.$$

It is clear that $\operatorname{Ker}\alpha^{-1}$ is thick and closed by isomorphisms. The objects of $\operatorname{Ker}\alpha^{-1}$ are called the ephemeral modules. These are the Alexandrov sheaves which vanishes when evaluated on a γ-open.

Proposition 4.19 ([BP18, Proposition 3.15])**.** *The category $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$ is the quotient of the category $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ by $\operatorname{Ker}\alpha^{-1}$ via the localization functor $\alpha^{-1}\colon \mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}}) \to \mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$. In particular, $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})/\operatorname{Ker}\alpha^{-1} \simeq \mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$.*

4.3.2 The interleaving distance

We start by defining certain morphisms canonically associated with Alexandrov and γ-sheaves that are used to define the interleaving distances.

Let $v,\ w \in \mathbb{V}$ and assume that $w \leq_\gamma v$. Let $F \in \operatorname{Mod}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$ (resp. $\operatorname{Mod}(\mathbf{k}_{\mathbb{V}_\gamma})$). Since $w + \gamma \subset v + \gamma$, it follows that for every $U \in \operatorname{Op}(\mathbb{V}_\mathfrak{a})$ (resp. $\operatorname{Op}(\mathbb{V}_\gamma)$), $U + w \subset U + v$. Hence, the restriction morphisms $\rho_{U+v,U+w}$ of F allows to define a morphism of sheaves

$$\chi^\bullet_{v,w}(F)\colon \tau_{v*}F \to \tau_{w*}F \tag{4.3.2}$$

by setting for every open subset U, $\chi^{\bullet}_{v,w}(F)_U := \rho_{U+v,U+w}$ with $\bullet = \mathfrak{a}$ if $F \in \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$ and $\bullet = \gamma$ if $F \in \mathrm{Mod}(\mathbf{k}_{\mathbb{V}_\gamma})$. This construction extends immediately to the derived category $\mathrm{D}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$ (resp. $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$).

We now construct similar morphisms for sheaves in $\mathrm{D}^{\mathrm{b}}_{\gamma^\circ,a}(\mathbf{k}_{\mathbb{V}})$. This construction is classical (see for instance [GS14]).

Lemma 4.20. *Let $F \in \mathrm{D}^{\mathrm{b}}_{\gamma^\circ,\, a}(\mathbf{k}_{\mathscr{V}})$ and $u \in \mathbb{V}$. Then there is a functorial isomorphism*

$$\tau_{u*}F \simeq \mathbf{k}_{-u+\gamma^a} \overset{\mathrm{np}}{\star} F.$$

For $w \leq_\gamma v$, the canonical map

$$\mathbf{k}_{-v+\gamma^a} \to \mathbf{k}_{-w+\gamma^a}$$

induces a morphism of functors

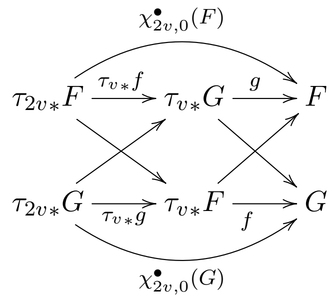
$$\mathbf{k}_{-v+\gamma^a} \overset{\mathrm{np}}{\star} (\cdot) \to \mathbf{k}_{-w+\gamma^a} \overset{\mathrm{np}}{\star} (\cdot). \tag{4.3.3}$$

Using Lemma 4.20, we obtain a morphism of functors from $\mathrm{D}^{\mathrm{b}}_{\gamma^\circ,\, a}(\mathbf{k}_{\mathbb{V}})$ to $\mathrm{D}^{\mathrm{b}}_{\gamma^\circ,\, a}(\mathbf{k}_{\mathbb{V}})$

$$\chi^{\mu}_{v,w} \colon \tau_{v*} \to \tau_{w*}. \tag{4.3.4}$$

Let $\mathcal{C}$ be any of the following categories $\mathrm{D}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$, $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$, $\mathrm{D}^{\mathrm{b}}_{\gamma^\circ,\, a}(\mathbf{k}_{\mathbb{V}_\gamma})$.

Definition 4.21. Let $F,\ G \in \mathcal{C}$, and $v \in \gamma^a$. We say that F and G are *v-interleaved* if there exists $f \in \mathrm{Hom}_{\mathcal{C}}(\tau_{v*}F, G)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(\tau_{v*}G, F)$ such that the following diagram commutes.



with $\bullet = \mathfrak{a}$ (resp. γ, μ) if $F, G \in \mathrm{D}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$ (resp. $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$, $\mathrm{D}^{\mathrm{b}}_{\gamma^\circ,\, a}(\mathbf{k}_{\mathbb{V}})$).

Definition 4.22. With the same notations, define the interleaving distance between F and G with respect to $v \in \gamma^a$ to be :

$$d^v_{I_\bullet}(F,G) := \inf(\{c \geq 0 \mid F \text{ and } G \text{ are } c \cdot v - \text{interleaved}\} \cup \{\infty\}).$$

where $\bullet = \mathfrak{a}$ (resp. γ, μ) if $F, G \in \mathrm{D}(\mathbf{k}_{\mathbb{V}_{\mathfrak{a}}})$ (resp. $\mathrm{D}(\mathbf{k}_{\mathbb{V}_\gamma})$, $\mathrm{D}^{\mathrm{b}}_{\gamma^\circ,\, a}(\mathbf{k}_{\mathbb{V}})$).

The interleaving distance was introduced in [CCSG+09] in the one dimensional case. One can use [dSMS18, Theorem 2.5] to see that $d_{I_\bullet}$ is a pseudodistance.

We obtain the following metric charaterization of the objects of $\operatorname{Ker}\alpha^{-1}$ which is similar to the result of [CCBdS16a] for persistence module over $\mathbb{R}$. Our approach and their approach are quite different as they rely on the structure theorem for 1-D persistent modules to establish their results. This structural result does not hold in the multipersistent setting.

Corollary 4.23 ([BP18, Corollary 4.17]). *Let $v \in \operatorname{Int}(\gamma^a)$. Then $F \in \operatorname{Ker}\alpha^{-1}$ if and only if $d^v_{I_\mathfrak{a}}(F, 0) = 0$.*

4.3.3 Isometry results

Interleaving distances

We obtain the following isometry results. They imply that the different formalization of the notion of persistence module are indiscernible from a metric point of view. This also implies that from an applied perspective, these formulations have the same discriminative power.

Theorem 4.24 ([BP18, Theorem 4.21]). *Let $v \in \operatorname{Int}(\gamma^a)$, F, $G \in \mathrm{D}(\mathbf{k}_{\mathbb{V}_\mathfrak{a}})$. Then :*

$$d^v_{I_\mathfrak{a}}(F, G) = d^v_{I_\gamma}(\beta_* F, \beta_* G).$$

Proposition 4.25. *The functor $\mathrm{R}\phi_{\gamma *} \colon \mathrm{D}^{\mathrm{b}}_{\gamma^\circ, a}(\mathbf{k}_\mathbb{V}) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}_\gamma})$ and its quasi inverse ϕ_γ^{-1} are isometries i.e.*

(i) for every F, $G \in \mathrm{D}^{\mathrm{b}}_{\gamma^\circ, a}(\mathbf{k}_\mathbb{V})$, $d^v_{I_\mu}(F, G) = d^v_{I^\gamma}(\mathrm{R}\phi_{\gamma *}F, \mathrm{R}\phi_{\gamma *}G)$,

(ii) for every F, $G \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}_\gamma})$, $d^v_{I_\gamma}(F, G) = d^v_{I_\mu}(\phi_\gamma^{-1}F, \phi_\gamma^{-1}G)$.

The convolution and the interleaving distances

In this subsection, we compare the interleaving distance on $\mathrm{D}^{\mathrm{b}}_{\gamma^\circ, a}(\mathbf{k}_\mathbb{V})$ with the convolution distance on $\mathrm{D}^{\mathrm{b}}_{\gamma^\circ, a}(\mathbf{k}_\mathbb{V})$. We sharpen Proposition 5.8 and Corollary 5.9 of [BP18] by removing the γ-properness assumption. The architecture of the proof is the same as in [BP18]. The γ-properness hypothesis is removed thanks to Theorem 4.1.

Here, $\mathbb{V}$ is endowed with a closed proper convex cone γ with non-empty interior. Let $v \in \operatorname{Int}(\gamma^a)$ and consider the set

$$B_v := (v + \gamma) \cap (-v + \gamma^a).$$

The set B_v is a symmetric closed bounded convex subset of $\mathbb{V}$ such that $0 \in \operatorname{Int} B_v$. It follows that the gauge

$$g_{B_v}(x) = \inf\{\lambda > 0 \,|\, x \in \lambda B_v\} \tag{4.3.5}$$

is a norm, the unit ball of which is B_v. We denote this norm by $\|\cdot\|_v$. From now on, we consider $\mathbb{V}$ equipped with this norm. Recall the map

$$u : \mathbb{V} \times \mathbb{V} \to \mathbb{V}, \quad (x, y) \mapsto x - y.$$

Consider

$$\Theta = \{(x, y) \in \mathbb{V} \times \mathbb{V} | x - y \in \gamma^a\} = u^{-1}(\gamma^a).$$

and notice that

$$\Delta_c = \{(x, y) \in \mathbb{V} \times \mathbb{V} | \, \|x - y\|_v \leq c\} = u^{-1}(B_c)$$

where B_c is the ball of center 0 and radius c in $\mathbb{V}$ for the norm $\|\cdot\|_v$.

The following formulas follow from Lemma 4.2.

$$\mathbf{k}_\Theta \overset{\mathrm{np}}{\circ} F \simeq \mathbf{k}_{\gamma^a} \overset{\mathrm{np}}{\star} F. \tag{4.3.6}$$

$$\mathbf{k}_{\Delta_c} \overset{\mathrm{np}}{\circ} \mathbf{k}_\Theta \simeq \mathbf{k}_{\Delta_c + \Theta}. \tag{4.3.7}$$

Lemma 4.26. *Let* $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_\mathbb{V})$*. Then*

$$(\mathbf{k}_{\Delta_c} \overset{\mathrm{np}}{\circ} \mathbf{k}_\Theta) \overset{\mathrm{np}}{\circ} F \simeq \mathbf{k}_{\Delta_c} \overset{\mathrm{np}}{\circ} (\mathbf{k}_\Theta \overset{\mathrm{np}}{\circ} F).$$

Proof. Let $\pi \colon T^*\mathbb{V} \to \mathbb{V}$ be the cotangent bundle of $\mathbb{V}$. We denote by u_d the dual of the tangent morphism of u. It fits in the following commutative diagram

$$\begin{array}{ccccc} T^*(\mathbb{V}\times\mathbb{V}) & \xleftarrow{u_d} & (\mathbb{V}\times\mathbb{V}) \times_\mathbb{V} \times T^*\mathbb{V} & \xrightarrow{u_\pi} & T^*\mathbb{V} \\ & \searrow^{\pi} & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ & & \mathbb{V}\times\mathbb{V} & \xrightarrow{u} & \mathbb{V}. \end{array}$$

Since, we have the isomorphism $\mathbf{k}_{\Delta_c} \simeq u^{-1}\mathbf{k}_{B_c}$, it follows from [KS90, Propostion 5.4.5] that $\mathrm{SS}(\mathbf{k}_{\Delta_c}) = u_d\, u_\pi^{-1}(SS(B_c))$. A direct computation shows that $u_d\, u_\pi^{-1}(T^*\mathbb{V}) \cap T^*_\mathbb{V}\mathbb{V} \times T^*\mathbb{V} \subset T^*_{\mathbb{V}\times\mathbb{V}}(\mathbb{V}\times\mathbb{V})$. Hence $\mathrm{SS}(\mathbf{k}_{\Delta_c}) \cap T^*_\mathbb{V}\mathbb{V} \times T^*\mathbb{V} \subset T^*_{\mathbb{V}\times\mathbb{V}}(\mathbb{V}\times\mathbb{V})$. Since $\mathbf{k}_{\Delta_c}$ is constructible and the properness assumption is clearly satisfied, we apply Theorem 4.1 and get the desired isomorphism. □

Finally, there is also the following isomorphism

$$\mathbf{k}_{c\cdot v+\gamma^a} \simeq \mathbf{k}_{B_c+\gamma^a}. \tag{4.3.8}$$

We now state and prove the sharpen version of [BP18, Proposition 5.8].

Theorem 4.27. *Let* $v \in \mathrm{Int}(\gamma^a)$*,* $c \in \mathbb{R}_{\geq 0}$ *and* $F, G \in \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ}, a}(\mathbf{k}_\mathbb{V})$*. Then* F *and* G *are* $c \cdot v$*-interleaved if and only if they are* c*-isomorphic.*

Proof. Let $F, G \in \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ},a}(\mathbf{k}_{\mathbb{V}})$. Assume they are $c \cdot v$-interleaved. We set $w = c \cdot v$. Hence, we have the maps

$$\alpha\colon \tau_{w*}F \to G \qquad\qquad \beta\colon \tau_{w*}G \to F$$

such that the below diagrams commute

$$\begin{tikzcd} \tau_{2w*}F \arrow[r, "\tau_{w*}\alpha"] \arrow[rr, bend right, "\chi^{\mu}_{0,2w}(F)"'] & \tau_{w*}G \arrow[r, "\tau_{w*}\beta"] & F \end{tikzcd} \qquad \begin{tikzcd} \tau_{2w*}G \arrow[r, "\tau_{w*}\beta"] \arrow[rr, bend right, "\chi^{\mu}_{2w,0}(G)"'] & \tau_{w*}F \arrow[r, "\tau_{w*}\alpha"] & G. \end{tikzcd}$$

Using Lemmas 4.20, we obtain

$$\begin{tikzcd} \mathbf{k}_{2w+\gamma^a} \overset{\mathrm{np}}{\star} F \arrow[r, "\mathbf{k}_{2w+\gamma^a} \overset{\mathrm{np}}{\star} \alpha"] \arrow[rr, bend right, "\chi_{2w,0} \overset{\mathrm{np}}{\star} F"'] & \mathbf{k}_{w+\gamma^a} \overset{\mathrm{np}}{\star} G \arrow[r, "\mathbf{k}_{w+\gamma^a} \overset{\mathrm{np}}{\star} \beta"] & F. \end{tikzcd}$$

Moreover for every $c \geq 0$, we have the following isomorphisms

$$\begin{aligned}
\mathbf{k}_{cv+\gamma^a} \overset{\mathrm{np}}{\star} F &\simeq \mathbf{k}_{B_c+\gamma^a} \overset{\mathrm{np}}{\star} F \qquad \text{by Equation (4.3.8)}\\
&\simeq \mathbf{k}_{\Delta_c+\Theta} \overset{\mathrm{np}}{\circ} F \qquad \text{by Lemma 4.2 (i)}\\
&\simeq (\mathbf{k}_{\Delta_c} \overset{\mathrm{np}}{\circ} \mathbf{k}_{\Theta}) \overset{\mathrm{np}}{\circ} F \qquad \text{by Equation (4.3.7)}\\
&\simeq \mathbf{k}_{\Delta_c} \overset{\mathrm{np}}{\circ} (\mathbf{k}_{\Theta} \overset{\mathrm{np}}{\circ} F) \qquad \text{by Lemma 4.26}\\
&\simeq \mathbf{k}_{\Delta_c} \overset{\mathrm{np}}{\circ} (\mathbf{k}_{\gamma^a} \overset{\mathrm{np}}{\star} F) \qquad \text{by Equation (4.3.6)}\\
&\simeq \mathbf{k}_{\Delta_c} \overset{\mathrm{np}}{\circ} F \qquad \text{by Proposition 4.17}\\
&\simeq \mathbf{k}_{B_c} \overset{\mathrm{np}}{\star} F \qquad \text{by Lemma 4.2 (i)}\\
&\simeq \mathbf{k}_{B_c} \star F \qquad \text{(compacity of } B_c).
\end{aligned}$$

Hence, we obtain the commutative diagram

$$\begin{tikzcd} \mathbf{k}_{B_{2c}} \star F \arrow[r, "\mathbf{k}_{B_c}\star\alpha"] \arrow[rr, bend right, "\rho_{0,2c}\star F"'] & \mathbf{k}_{B_c} \star G \arrow[r, "\beta"] & F\ . \end{tikzcd}$$

Similarly we obtain the following commutative diagram

$$\begin{tikzcd} \mathbf{k}_{B_{2c}} \star G \arrow[r, "\mathbf{k}_{B_c}\star\beta"] \arrow[rr, bend right, "\rho_{0,2c}\star G"'] & \mathbf{k}_{B_c} \star F \arrow[r, "\alpha"] & G \end{tikzcd} .$$

Hence, F and G are c-isomorphic.

A similar argument proves that if F and G are c-isomorphic then they are $c \cdot v$-interleaved. $\square$

Corollary 4.28. *Let* $v \in \operatorname{Int} \gamma^a$, $F, G \in \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ},a}(\mathbf{k}_{\mathbb{V}})$. *Then*

$$\operatorname{dist}_{\mathbb{V}}(F,G) = d^{v}_{I^{\mu}}(F,G)$$

where $\operatorname{dist}_{\mathbb{V}}$ *is the convolution distance associated with the norm* $\|\cdot\|_v$.

Chapter 5

Projects and perspectives

My current research is in a large part concerned with the development of the theoretical foundation of multi-persistence by relying on the methods of microlocal sheaf theory and the application of topological data analysis to precision medicine whose aim is to tailor treatments according to patients' characteristics. I also started to work on problems of causal inference related to the estimation of the heterogeneous and average treatment effects. Below, I list some of the projects I am working on or intend to work on.

- With Nicolas Berkouk, we are pursuing our study of the metric aspects of TDA. We are currently working on Conjecture 4.15 (We refer to subsection 4.2.5 and to the paragraph *A conjecture* for more details) and intend to compare various distances (convolution, erosion, weighted bottleneck...) for multipersistence modules. We are also interested in distances on constructible functions and their relations with distances on constructible sheaves either via the "faisceaux–fonctions" correspondence or via generalizations of the rank invariant for persistence modules that we are presently constructing.

- With Steve Oudot, we aim at designing new computable invariants and vectorizations for multipersistence modules. Our strategy is to use operations on sheaves (for instance pushforward by suitable families of functions) to turn multi-parameter persistence modules into various families of one-parameter modules. This will then allow us to reuse some of the machinery from one-parameter persistence to build our invariants and their vectorizations, in the same spirit as (yet in a much more refined way than) fibered barcodes, which are currently one of the few invariants for multi-parameter TDA.

- With François-Camille Grolleau, Raphaël Porcher and Steve Oudot, we aim at integrating topological features associated with longitudinal data in the design of optimal dynamic treatment regimen (DTR), to better take into account the global structure of the data. We will focus on the development of a DTR for intensive care unit patients with an acute kidney injury. As there are widely accepted clinical criteria regarding when to cease the dialysis of such patients, we will focus on determining the optimal starting time of the dialysis. The construction of the topological features to include in the DTR and the selection of their parameters raises several theoretical issues, which can be naturally approached via multipersistence theory.

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